AN INDETERMINATE PROBLEM IN CLASSICAL MECHANICS

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1. Introduction. We shall say that a physical experiment leads to an indeterminant result if upon reproducing the conditions of the experiment one obtains differing results, and this scatter is independent of the accuracy with which the experiment is conducted. In modern atomic physics one often performs experiments in which the exact outcome is indeterminate and the theory only predicts the probability of the occurrence of each possible outcome. We shall show here that such experiments may also be devised within the realm of the purely classical mechanics of collisions of particles and, in fact, that one of the popular freshman laboratory demonstrations, when slightly modified, leads to an indeterminate problem.

Anyone who has taken an elementary physics course has probably been exposed to the following demonstration. A set of balls is constrained to move in a straight line, perhaps by means of a long narrow trough in which they are free to roll. (To avoid rotation the experiment may also be performed with beads strung on a straight, stiff wire.) Initially m of the balls are in contact and at rest while the remaining n balls are also in contact but are moving with a velocity v toward the stationary balls. After collision one finds that the n balls at the front of the line are moving forward with velocity v, while the m balls at the back are now stationary. For this experiment it is necessary that all balls have the same mass. How does one account for this result? Well, clearly the laws of conservation of momentum and energy are satisfied. However, in the case of more than two balls there will be an infinite number of outcomes satisfying these conditions. For example, suppose there are three balls each with mass 1 and suppose that the first moves with velocity 1 toward the second and third which are in contact and stationary. Denoting the final velocities of the balls by v_1, v_2, and v_3, the following conditions must be satisfied:

$$
\text{(I)}\qquad
\begin{aligned}
&\text{(a)}\quad v_1 + v_2 + v_3 = 1 \quad \text{(conservation of momentum)},\\
&\text{(b)}\quad v_1^2 + v_2^2 + v_3^2 = 1 \quad \text{(conservation of energy)},\\
&\text{(c)}\quad v_1 \leqq v_2 \leqq v_3 \qquad \text{(no ball can overtake another)}.
\end{aligned}
$$

Now consider the solutions of (I) as a subset of three-space. The intersection of (a) and (b) is a circle on the surface of the unit sphere while (c) determines a wedge shaped region, so the solutions form an arc of a circle. The actual solution is of course $v_1=0$, $v_2=0$, $v_3=1$, but another possibility would be, for instance, $v_1=-1/3$, $v_2=2/3$, $v_3=2/3$.

We might try to account for the result by a continuity argument. That is, let the balls be separated by a finite amount, analyze the successive two ball collisions, and take the limit as the distances between balls go to zero. It is easy to see that in the case of the classical demonstration described above this limit actually exists, for in each collision a ball gives up all its velocity to a

stationary ball. On the other hand, suppose the balls have different velocities or different masses. The following examples show that the continuity argument may break down completely, and we shall try to convince the reader that the outcome of these experiments will be indeterminate.

2. First example. Given three balls constrained to move on a straight line, the first with mass 1, the second with mass 3, and the third with mass 1. Initially the second ball is stationary while the first and third move toward it with unit speed from equal distances. From considerations of symmetry we would expect that after collision the center ball would remain stationary while the ones on the outside would move apart with unit speed. Suppose, however, that one of the balls, say the first one, strikes the center ball slightly before the third. Let us analyze the two ball collisions that will then occur. Recall that the outcome of a two particle straight line elastic collision is completely determined by the laws of conservation of energy and momentum. If the particles have masses m_1 and m_2 and the velocities before impact are u_1 and u_2, then the velocities after impact must satisfy the pair of equations:

$$\text{(II)} \qquad \begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ m_1u_1^2 + m_2u_2^2 &= m_1v_1^2 + m_2v_2^2. \end{aligned}$$

This pair of equations has two solutions, the trivial one $v_1 = u_1$ and $v_2 = u_2$, when no collision occurs, and the second solution in which we are interested,

$$\text{(III)} \qquad v_1 = \frac{2m_2u_2 - (m_2 - m_1)u_1}{m_1 + m_2}, \qquad v_2 = \frac{2m_1u_1 - (m_2 - m_2)u_2}{m_1 + m_2}.$$

Let us now apply (III) to the three ball situation described above. Denoting the velocities of the first, second and third balls by the subscripts 1, 2, and 3, we find that after the first collision the first ball attains a velocity $v_1 = -1/2$ and the second ball $u_2 = 1/2$. The second ball then hits the third ball and after collision the velocities are $v_2 = -1/4$ and $v_3 = 5/4$. The final velocities are thus,

$$\text{(IV)} \qquad v_1 = -1/2, \qquad v_2 = -1/4, \qquad v_3 = 5/4.$$

On the other hand, if the third ball strikes the second ball first the final speeds will be

$$\text{(IV)}' \qquad v_1 = -5/4, \qquad v_2 = 1/4, \qquad v_3 = 1/2.$$

Thus we see that the final result depends discontinuously on the initial positions of the balls. In the critical case where both balls hit simultaneously symmetry considerations give an answer, but it is clear that this symmetry can be upset by slightly altering the masses or initial velocities of balls one or three. What then should we expect as the outcome of our experiment? The only answer seems to be that no matter how we refine the experimental technique we may get either case (IV) or (IV)′ and thus our result will be indeterminate in the sense defined above.

The reader acquainted with mechanics may at this point raise an objection. In adopting the second of equations (II) we are making the assumption that the colliding bodies are "perfectly elastic" and hence no energy is lost in collision. In practice there is no such thing as a perfectly elastic body. Is it not this idealization which gives rise to the discontinuity? The answer is that even without the assumption of perfect elasticity the indeterminate result still holds, as we now show.

For the case of non-perfect elasticity equations (II) are replaced by,

$$\text{(V)} \qquad \begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ v_1 - v_2 &= -e(u_1 - u_2), \qquad \text{where } 0 \leqq e \leqq 1. \end{aligned}$$

The quantity e is called the coefficient of restitution and depends on the material from which the particles are made. (For derivation of these equations see texts on particle mechanics, for instance, Lindsay, *Physical Mechanics*, pp. 350.) These equations become equivalent to (II) in the case where $e=1$. Equations (III) are now replaced by:

$$\text{(VI)} \qquad \begin{aligned} v_1 &= \frac{(m_1 - em_2)u_1 + m_2(1 + e)u_2}{m_1 + m_2}, \\ v_2 &= \frac{m_1(1 + e)u_1 + (m_2 - m_1e)u_2}{m_1 + m_2}. \end{aligned}$$

Consider now the above experiment where balls 1 and 3 have mass m_1 and ball 2 has mass m_2. Again assume that 2 is struck by 1 before it is struck by 3. Using (VI) we find that after the two collisions we get as final velocities,

$$\text{(VII)} \qquad \begin{aligned} v_1 &= \frac{m_1 - em_2}{m_1 + m_2}, \qquad v_2 = -\left(\frac{m_1(1 + e)}{m_1 + m_2}\right)^2, \\ v_3 &= \frac{m_1m_2(1 + e)^2 - (m_1 + m_2)(m_1 - m_2e)}{(m_1 + m_2)^2}. \end{aligned}$$

If we chose m_2 so large that $n_1 < em_2$, then $v_1<0$, $v_2<0$, and $v_3>0$. To guarantee that there will be no further collisions we must also know that $v_1<v_2$, which can also be achieved by taking m_2 sufficiently large, since

$$\lim_{m_2\to\infty} v_1 = -e \quad \text{and} \quad \lim_{m_2\to\infty} v_2 = 0.$$

It is now clear that if our collisions occur in the reverse order we obtain

$$\text{(VII)}' \qquad v_1' = -v_3, \qquad v_2' = -v_2, \qquad v_3' = -v_1.$$

Thus in one case v_2 is negative, in the other positive, and we again observe the discontinuity and hence the indeterminacy.

We have shown that the simultaneous collision of three particles may lead to an indeterminate result. We shall now use this to give an example which

shows that the classical experiment with four balls may lead to an indeterminacy if the masses of the balls are suitably selected. To avoid having to work with long complicated formulae we shall pick definite numerical values for the masses of the balls. Also we shall for simplicity assume perfect elasticity although, in view of the previous example, it will be clear that this assumption is not necessary.

3. Second example. We are given four balls, B_1, B_2, B_3, and B_4, with masses 2, 1, 1, and 3 respectively. Balls 2, 3, and 4 are in contact and stationary, while ball 1 moves towards them with velocity 1. Let u, v, w, and x be the final velocities of the balls.

Assertion: The experiment may lead to one of two possible results.

Case I. $u=-1/3$, $v=0$, $w=2/3$, $x=2/3$.

Case II. $u=-11/27$, $v=10/27$, $w=4/9$, $x=2/3$.

Proof: Suppose balls 2 and 3 and balls 3 and 4 are separated by a distance ϵ. We must now analyze a series of two ball collisions which we shall number in chronological order. We assume that the first collision occurs at time $t_1=0$, and that the i-th collision occurs at time t_i. The symbols u_i, v_i, w_i, and x_i will denote the respective velocities after the i-th collision. Initially we have

$$u_0 = 1, \qquad v_0 = 0, \qquad w_0 = 0, \qquad x_0 = 0.$$

Collision 1. B_1 strikes B_2 at time $t_1=0$. Then

$$u_1 = 1/3, \qquad v_1 = 4/3, \qquad w_1 = 0, \qquad x_1 = 0.$$

Collision 2. B_2 strikes B_3 at time $t_2=\epsilon/(4/3)=3\epsilon/4$. Then

$$u_2 = 1/3, \qquad v_2 = 0, \qquad w_2 = 4/3, \qquad x_2 = 0.$$

We must now determine which collision will occur next, between balls 1 and 2 or balls 3 and 4. Since B_1 is now distance ϵ from B_2, and B_3 is distance ϵ from B_4, it is clear that 3 will strike 4 first since $w_2>u_2$.

Collision 3. B_3 strikes B_4 at $t_3=t_2+3\epsilon/4=3\epsilon/2$. Then

$$u_3 = 1/3, \qquad v_3 = 0, \qquad w_3 = -2/3, \qquad x_3 = 2/3.$$

We must now determine whether B_1 or B_3 will strike B_2 first. If B_1 strikes first, the collision occurs at time $t_4=t_1+\epsilon/(1/3)=3\epsilon$. If B_3 strikes first, the collision occurs at $t_4=t_3+\epsilon/(2/3)=3\epsilon$.

Conclusion: Both collisions occur at the same time! Thus we are again in an indeterminate situation similar to that of the first example and must resort to analyzing two separate cases according as B_3 or B_1 hits B_2 first.

Case I.

Collision 4. Between B_3 and B_2. Then

$$u_4 = 1/3, \qquad v_4 = -2/3, \qquad w_4 = 0, \qquad x_4 = 2/3.$$

Collision 5. Between B_2 and B_1. Then

$$u_5 = -1/3, \quad v_5 = 2/3, \quad w_5 = 0, \quad x_5 = 2/3.$$

Collision 6. Between B_2 and B_3. Then

$$u_6 = -1/3, \quad v_6 = 0, \quad w_6 = 2/3, \quad x_6 = 2/3.$$

Since now $u \leqq v \leqq w \leqq x$, there will be no more collisions.

Case II.

Collision 4′. Between B_1 and B_2.

$$u_{4'} = 1/9, \quad v_{4'} = 4/9, \quad w_{4'} = -2/3, \quad x_{4'} = 2/3.$$

Collision 5′. Between B_2 and B_3.

$$u_{5'} = 1/9, \quad v_{5'} = -2/3, \quad w_{5'} = 4/9, \quad x_{5'} = 2/3.$$

Collision 6′. Between B_1 and B_2.

$$u_{6'} = -11/27, \quad v_{6'} = 10/27, \quad w_{6'} = 4/9, \quad x_{6'} = 2/3. \quad \text{Q.E.D.}$$

(The skeptical reader is encouraged to check the author's calculations by verifying that the two conservation laws are satisfied at each collision.)

The above example illustrates another surprising fact. If we consider the case of the separated balls we have seen that the final velocities are discontinuous functions of the initial positions of the balls. That is, suppose that initially the distance between B_1 and B_2 is ϵ_{12}, and the distance between B_3 and B_4 is ϵ_{34}. Then we will get case I or case II according as $\epsilon_{23} > \epsilon_{34}$ or $\epsilon_{34} > \epsilon_{23}$, while the critical case, $\epsilon_{23} = \epsilon_{34}$, is indeterminate. It is also true that the final velocities are discontinuous functions of the masses of the balls. We leave it to the reader to convince himself that if $m_4 > 3$ we get case I, while if $m_4 < 3$ we get case II.

4. Summary. Let us try to summarize the significance of our results. We have seen that in what we have called the classical collision experiment, the usual laws of collision (with or without perfect elasticity) are insufficient to determine a unique outcome. There are then two alternative possibilities: first, that there exist some further physical laws which together with the classical ones will determine the result uniquely; second, that the indeterminacy actually exists in nature and that the experiment if repeated would give differing results. We have tried to argue that it is, in fact, the second alternative which holds. If allowed a slightly metaphysical comment we would put it this way. Philosophers talk of the law of "uniformity of nature" which states that given the same initial conditions the same results will follow. This "law" has had to be abandoned in the field of atomic physics. We suggest here that it may also fall down in the classical case.

ROBINSON'S CONSTANT

Editorial Note. A recent issue of this MONTHLY (Volume 58, No. 7, August-September, 1951) contained an article *A New Absolute Geometric Constant?* by Robin Robinson. This paper is a summary of the articles submitted to the editor dealing with the value of this constant.

It was proved that the number g_n of polygons of n sides (including "degenerate" cases) formed by a network of n lines (no three concurrent) is given by

$$g_{n+1} = ng_n + \binom{n}{2} g_{n-2},$$

where $g_1 = g_2 = 0$, $g_3 = 1$. Introducing u_n, where

$$g_n = u_n \cdot \tfrac{1}{2}(n-1)!,$$

this provides the recursion relation

$$u_{n+3} = u_{n+2} + \frac{1}{2n} u_n, \tag{1}$$

where $u_1 = u_2 = 0$, $u_3 = u_4 = u_5 = 1$. The author proved that

$$\lim_{n\to\infty} \frac{u_n^2}{n} = b,$$

where b is the constant 0.284,098, $\cdots$, and expressed an interest in finding a finite expression for b.

Thirteen readers submitted evidence that Robinson's constant b is given by

$$b = 4e^{-3/2}/\pi.$$

These readers were David Dickinson, Rufus Isaacs, Harry Pollard, R. M. Redheffer, John Riordan, H. F. Sandham, M. R. Spiegel, C. S. Sutton, G. Szekeres, F. Ursell, Morgan Ward, G. T. Williams, and Koichi Yamamoto.

The methods used were varied. Most treatments made use of a generating function, the one most commonly used being

$$f(x) = 2\sum_{n=3}^{\infty} \frac{g_n}{n!} x^n = \sum_{n=3}^{\infty} \frac{u_n}{n} x^n.$$

This series converges for $|x| < 1$. By use of the relation (1) it is readily shown that

$$x^2[f(x) + 2] = 2(1 - x)f'(x).$$

Since $f(0) = 0$, this leads to

$$f(x) + 2 = 2(1 - x)^{-1/2} \exp\left(-\frac{x^2 + 2x}{4}\right). \tag{2}$$

Yamamoto also derived this generating function directly from consideration of the geometric problem, without use of any recursion relation.

From this point on the various treatments diverged widely. Probably the neatest treatment (used by Redheffer, Ursell, and Williams), made use of the following theorem (see Titchmarsh, *Theory of Functions*, 2nd edition, p. 225):

If $f(x)=\sum_0^\infty a_n x^n$, $|x|<1$, and $g(x)=\sum_0^\infty b_n x^n$, $|x|<1$, where $\sum_0^\infty b_n$ diverges, and if $\lim_{n\to\infty}\{a_n/b_n\}=C$, then $\lim_{x\to 1}\{f(x)/g(x)\}=C$.

Applying this theorem to the generating function $f(x)$, with $g(x)=(1-x)^{-1/2}$, and using Stirling's formula to simplify the binomial coefficients b_n, we obtain

$$\lim_{x\to 1}\frac{f(x)}{(1-x)^{-1/2}}=\lim_{n\to\infty}\frac{u_n}{n}\sqrt{\pi n}$$

$$=\sqrt{\pi}\lim_{n\to\infty}\frac{u_n}{\sqrt{n}}=\sqrt{\pi b}.$$

But from (2) this limit also equals $2e^{-3/4}$, whence

$$b=4e^{-3/2}/\pi.$$

Robinson's other constant $B=\sqrt{\frac{1}{2}\pi b}$ is then $\sqrt{2}e^{-3/4}$, and

$$g_n\cong\sqrt{2}\,n^n e^{-n-3/4}.$$

The one treatment which had little resemblance to the foregoing was by Isaacs, and made ingenious use of contour integration. He set up the integral

$$u_n=\frac{e^{-3/4}}{\pi i}\int\frac{\exp(z^2-\frac{1}{4}z^4)}{z^2(1-z^2)^{n-2}}\,dz, \tag{3}$$

where the path of integration is a small circle taken *clockwise* about $z=1$. That (3) satisfies (1) is shown by substitution including an integration by parts. It is obvious that $u_1=u_2=0$; by computation of a residue, $u_3=1$. From (1) it follows that $u_4=u_5=1$. Thus an explicit formula for u_n is provided. By careful deformation of the path, in conjunction with suitable changes of variable, Isaacs showed that as $n\to\infty$ the integral in (3) becomes infinite with $2i\sqrt{n\pi}$. Hence the result $b=4e^{-3/2}/\pi$. This treatment is independent of Robinson's proof of the existence of the limit b.

Yamamoto went on to discuss other questions, verifying some of Robinson's conjectures regarding the behavior of $u_n^2/(n-i)$ for various constants i. He also pointed out that a similar recursion relation and similar asymptotic behavior occur in an apparently unrelated setting in Problem 45, page 110, of Pólya-Szegö: *Aufgaben und Lehrsätze* II, Berlin 1925.

AN ELEMENTARY NOTE ON POWERS OF QUATERNIONS*

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1. A table of powers† of $z=x_0+i\rho$ can be readily used to obtain powers of quaternions of the form $X=x_0+x_1i+x_2j+x_3k$ whenever $x_1^2+x_2^2+x_3^2=\rho^2>0$, in virtue of the following result: If $z^n=a+ib$, then $X^n=a+\lambda(x_1i+x_2j+x_3k)$ where $\lambda=b/\rho$.

For example, to find $(7+8i+j+4k)^{13}$, since $8^2+1^2+4^2=9^2$, we look up $(7+9i)^{13}$ in the table of powers of z and find its real part to be 4065 65435 48992 and its imaginary part to be $-3709\ 17313\ 23456$. Then $(7+8i+j+4k)^{13}=4065\ 65435\ 48992-3709\ 17313\ 23456\cdot(8i+j+4k)/9=4065\ 65435\ 48992-412\ 13034\ 80384\ (8i+j+4k)$.

2. The simplest and most direct demonstration of the above relation between z^n and X^n was noted by Dr. E. W. Cannon, as follows: Let $X=x_0+x_1i+x_2j+x_3k=x_0+\vec{x}$, where $\vec{x}=x_1i+x_2k+x_3k$ and $\rho^2=x_1^2+x_2^2+x_3^2=-\vec{x}\,\vec{x}$. By quaternion multiplication, $X^2=x_0^2-\rho^2+2x_0\vec{x}$. This suggests a possible parallel with the formation of powers of the complex number $x_0+i\rho$. In the cases $n=1$ and $n=2$ we see: (A) if $(x_0+i\rho)^n$ is designated by $a+ib$, then $X^n=a+(b/\rho)\vec{x}$. Assume the truth of (A) for some particular n. Then from the rules of quaternion multiplication,

$$X^{n+1}=\left(a+\frac{b}{\rho}\,\vec{x}\right)(x_0+\vec{x})=(ax_0-b\rho)+\frac{a\rho+bx_0}{\rho}\,\vec{x}.$$

But

$$(x_0+i\rho)^{n+1}=(a+ib)\cdot(x_0+i\rho)=(ax_0-b\rho)+(a\rho+bx_0)i.$$

This establishes relation (A) for $n+1$, and thus demonstrates it for all positive integers n.

3. The relation between z^n and X^n mentioned above can also be explained in the following way from more general considerations (noted by the writer): Every power X^n of a quaternion X is commutative with X.‡ On the other hand, all quaternions Y which commute with X are of the form $\mu+\lambda X$ where μ and λ are real numbers.§ For, let

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† The most extensive is the recent National Bureau of Standards, Applied Mathematics Series no. 8, Table of Powers of Complex Numbers, which gives the exact values of the first twenty-five powers of $z=x_0+i\rho$ for x_0 and ρ each ranging from 0 through 10 at unit intervals.

‡ In connection with powers of quaternions see I. Niven, The roots of a quaternion, this MONTHLY, vol. 49, 1942, pp. 386–388, and L. Brand, The roots of a quaternion, this MONTHLY, vol. 49, 1942, pp. 519–520.

§ This demonstration that $Y=\mu+\lambda X$ is a slight modification of the writer's original proof, which is due to the referee who noticed that the original proof was incomplete.

$$X = x_0 + x_1 i + x_2 j + x_3 k, \quad \text{and}$$
$$Y = y_0 + y_1 i + y_2 j + y_3 k.$$

The assumption $XY = YX$ can be expanded in terms of the x_r and the y_r, and the equating of coefficients of i, j, and k gives

$$x_2 y_3 = x_3 y_2, \qquad x_1 y_3 = x_3 y_1, \qquad x_1 y_2 = x_2 y_1.$$

Since not all of x_1, x_2, x_3 are zero, there is no loss of generality in assuming that $x_1 \neq 0$. Thus we have

$$Y = y_0 + y_1 i + y_2 j + y_3 k = \left(y_0 - \frac{y_1 x_0}{x_1}\right) + \frac{y_1}{x_1} X.$$

Conversely, it is obvious that $XY = YX$ when Y has the form $\mu + \lambda X$.

To establish the relation in the first paragraph, we know that if $X^n = a_n + \lambda_n(x_1 i + x_2 j + x_3 k)$ and $X^{n+1} = a_{n+1} + \lambda_{n+1}(x_1 i + x_2 j + x_3 k)$, then

$$\begin{cases} a_{n+1} = x_0 a_n - \rho^2 \lambda_n \\ \lambda_{n+1} = a_n + x_0 \lambda_n \end{cases} \tag{1}$$

with $a_0 = 1$, $\lambda_0 = 0$, or $a_1 = x_0$, $\lambda_1 = 1$. Relations (1) show that a_n and λ_n depend only upon x_0 and ρ, and by induction a_n and λ_n are seen to be identical functions of x_0 and ρ whether we start from the quaternion $X = x_0 + x_1 i + x_2 j + x_3 k$ or the special case of the complex number $z = x_0 + i\rho$. Thus in X^n, the a_n is the same as the corresponding a_n in z^n which is a, and the λ_n is the same as the corresponding λ_n in z^n which is b/ρ. This is the result stated in the first paragraph. Thus from the NBS *Table of Powers of Complex Numbers*, we can find the powers of all quaternions $x_0 + x_1 i + x_2 j + x_3 k$ where $x_0 = 0(1)10$ and (x_1, x_2, x_3), considered as points in three-space, lie upon any of the spheres with center at the origin, of radius 0, 1, 2, $\cdots$, 10.

4. The following alternative demonstration of the above relation between z^n and X^n which explains why it holds from even more general considerations, is due to Olga Taussky: The quaternion $X = x_0 + x_1 i + x_2 j + x_3 k$ has the well-known representation by the matrix

$$X = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

whose characteristic roots are $r = x_0 \pm \sqrt{-x_1^2 - x_2^2 - x_3^2}$. Since X satisfies a quadratic equation, $X^n = \mu + \lambda X$. Thus X^n, written as $a_n + b_1 i + b_2 j + b_3 k$ can be written as $\mu + \lambda x_0 + \lambda(x_1 i + x_2 j + x_3 k)$. But the characteristic roots of X^n are $r^n = a \pm ib$. Let r be denoted by $z = x_0 \pm i\rho$, $\rho^2 = x_1^2 + x_2^2 + x_3^2$. Hence $r^n = a \pm ib = \mu + \lambda x_0 \pm \sqrt{-\lambda^2(x_1^2 + x_2^2 + x_3^2)} = \mu + \lambda x_0 \pm i\lambda\rho$, or $a_n = \mu + \lambda x_0 = a$, and $\lambda = b/\rho$.

5. To generate successive powers of quaternions beyond the range of available tables of z^n, one might use (1) or a simple modification which is given below in (5) and (6), which are obtained from (1) by eliminating the a's and λ's. In-

cidentally, (5) and (6) might also be found more suitable for calculating successive powers of ordinary complex numbers, especially for the higher powers where one might wish to avoid cross-multiplications of real and imaginary parts.

From (1), we obtain

$$a_{n+1}\lambda_n - a_n\lambda_{n+1} = -\lambda_n^2\rho^2 - a_n^2, \tag{2}$$

$$a_{n+1} - x_0\lambda_{n+1} = -(\rho^2 + x_0^2)\lambda_n, \quad \text{and} \tag{3}$$

$$x_0a_{n+1} + \rho^2\lambda_{n+1} = (\rho^2 + x_0^2)a_n. \tag{4}$$

Formula (2) need be considered only as a checking relationship. From (1) and

$$a_n = x_0\lambda_n - (\rho^2 + x_0^2)\lambda_{n-1}, \tag{3'}$$

we have

$$\lambda_{n+1} = 2x_0\lambda_n - (\rho^2 + x_0^2)\lambda_{n-1}. \tag{5}$$

Also, from (1) and

$$x_0a_n + \rho^2\lambda_n = (\rho^2 + x_0^2)a_{n-1}, \tag{4'}$$

we have

$$a_{n+1} = 2x_0a_n - (\rho^2 + x_0^2)a_{n-1}. \tag{6}$$

Thus (5) and (6) lead immediately to the continued fraction

$$p_0 + \cfrac{p_1}{q_1 + \cfrac{p_2}{q_2 + \cfrac{p_3}{q_3 + \cdots}}}$$

where $p_0 = x_0$, $p_n = -(\rho^2 + x_0^2)$, $q_n = 2x_0$, for $n = 1, 2, 3, \cdots$, of which a_n/λ_n is the n^{th} convergent, considering $x_0/1$ as the first convergent.

ON CERTAIN DETERMINANTAL EQUATIONS*

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1. Introduction. In many applied problems it is necessary to know the nature of the roots of equations which are determinantal in form. It is the purpose of this paper to prove a number of known results in an extremely simple way by continued fractions with the use of the criteria of Grommer [3] and Frank [1], and to make these results more widely available. Theorems are also developed which extend known results to determinantal equations with complex elements and with fewer non-zero elements (cf. Theorems 3.2, 3.3, 4.6).

* Presented to the American Mathematical Society, September 4, 1951.

2. The equation $P(z)=0$. In this section the equation

$$P(z)=\begin{vmatrix} z & a_2 & 0 & \cdots & 0 & 0 \\ -c_1 & z & a_3 & \cdots & 0 & 0 \\ 0 & -c_2 & z & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z & a_n \\ 0 & 0 & 0 & \cdots & -c_{n-1} & z \end{vmatrix}=0 \tag{2.1}$$

is studied.

Recently M. J. Moore [6] proved the following theorem concerning equation (2.1) by the use of induction on a series of similar determinantal equations. It is now proved by continued fractions. The proof is carried out in detail since all subsequent theorems can be proved in a similar way by continued fractions.

THEOREM 2.1. *The equation* (2.1), *where* $a_j>0$, $c_j>0$, $j=1, 2, \cdots, n$, *has* n *pure imaginary roots if* n *is even, and one root zero and* $(n-1)$ *pure imaginary roots if* n *is odd.*

Proof. This proof is based on the expansion of a rational function into a continued fraction

$$\frac{f_1}{f_0}=\frac{1}{r_1z+s_1+}\,\frac{1}{r_2z+s_2+}\,\cdots+\frac{1}{r_nz+s_n},$$

where

$$f_0=a_{00}z^n+a_{01}z^{n-1}+\cdots+a_{0n},$$
$$f_1=a_{11}z^{n-1}+a_{12}z^{n-2}+\cdots+a_{1n},$$

are two polynomials of degrees n and $n-1$, respectively. Such an expansion is possible if and only if

$$D_j\neq 0, \qquad j=0, 1, \cdots, n, \tag{2.2}$$

where $D_0=a_{00}$, and $D_1, D_2, \cdots, D_n$, are the first n principal minors of odd order (blocked off by lines) in the array

$$\begin{array}{|c|cc|cc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16}\cdots \\ \hline a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05}\cdots \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\cdots \\ \hline 0 & a_{00} & a_{01} & a_{02} & a_{03} & a_{04}\cdots \\ 0 & 0 & a_{11} & a_{12} & a_{13} & a_{14}\cdots \\ \hline 0 & 0 & a_{00} & a_{01} & a_{02} & a_{03}\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

where $a_{0j}=a_{1j}=0$ if $j>n$ (cf. [1], p. 147).

For n even, $n=2m$, equation (2.1) may be written in the form

$$P(z) = z^{2m} + \sum c_1a_2z^{2m-2} + \sum c_1a_2c_3a_4z^{2m-4} + \cdots + c_1a_2 \cdots c_{2m-1}a_{2m} = 0,$$

in which each summation represents the sum of all possible products of the type $c_1a_2c_3a_4 \cdots$ with ascending subscripts and with the c's and a's occurring alternately. Let $y=z^2$. Then

$$P(y) = y^m + \sum c_1a_2y^{m-1} + \sum c_1a_2c_3a_4y^{m-2} + \cdots + c_1a_2 \cdots c_{2m-1}a_{2m} = 0.$$

Form

$$P'(y) = my^{m-1} + (m-1)\sum c_1a_2y^{m-2} + \cdots + c_1a_2 \cdots c_{2m-3}a_{2m-2}.$$

The continued fraction expansion for $P'(y)/P(y)$ exists since the determinants D_j (2.2) are different from zero. The expansion may be obtained from the numbers in the following table by the algorithm given in [1] (table (2.4)):

$$a_{00} = 1 \qquad a_{01} = \sum c_1a_2 \qquad a_{02} = \sum c_1a_2c_3a_4 \qquad \cdots$$

$$a_{11} = m \qquad a_{12} = (m-1)\sum c_1a_2 \qquad a_{13} = (m-2)\sum c_1a_2c_3a_4 \cdots$$

$$b_{11} = \frac{\sum c_1a_2}{m} \qquad b_{12} = \frac{2\sum c_1a_2c_3a_4}{m} \qquad b_{13} = \frac{3\sum c_1a_2c_3a_4c_5a_6}{m} \qquad \cdots$$

$$a_{22} = \frac{2\sum c_1a_2c_3a_4 - \dfrac{(m-1)(\sum c_1a_2)^2}{m}}{m} \cdots$$

. . . . . .

$$r_1 = \frac{a_{00}}{a_{11}} = \frac{1}{m}, \quad s_1 = \frac{b_{11}}{a_{11}} = \frac{\sum c_1a_2}{m^2}, \quad r_j = \frac{a_{j-1,j-1}}{a_{j,j}}, \quad s_j = \frac{b_{j,j}}{a_{j,j}}, \quad j = 2, 3, \cdots, m.$$

Then

$$\frac{P'(y)}{P(y)} = \frac{1}{r_1y + s_1 + \dfrac{1}{r_2y + s_2 + \cdots + \dfrac{1}{r_my + s_m}}}, \tag{2.3}$$

where $r_1, r_3, \cdots$, are positive, and $r_2, r_4, \cdots$, are negative. Hence by the Grommer Theorem (cf. [2], Theorem 4.1, or [3]), all the roots of $P(y)=0$ are *real*. If $P(y)=0$ has multiple roots, $P'(y)$ and $P(y)$ have common factors which may first be removed by the euclidean algorithm.

The fact that all the roots of $P(y)=0$ are negative may be ascertained by a theorem of Frank (cf. [1], Theorem 4.1), if one forms

$$Q(y) = \sum c_1a_2y^{m-1} + \sum c_1a_2c_3a_4c_5a_6y^{m-3} + \cdots,$$

and then the expansion

$$(2.4)\qquad \frac{Q(y)}{P(y)} = \cfrac{1}{d_1 y + k_1 + 1 + \cfrac{1}{d_2 y + k_2 + \cdots + \cfrac{1}{d_m y + k_m}}}$$

by use of the algorithm (5.3) of [2]. It is found here that all the $d_j, j=1, 2, \cdots, m$, are *positive*, in fact, $d_1 = 1/\sum c_1 a_2$, $d_2 = (\sum c_1 a_2)^2 / [\sum c_1 a_2 \sum c_1 a_2 c_3 a_4 - \sum c_1 a_2 c_3 a_4 c_5 a_6]$, and $k_j = 0$. Consequently, all the roots of $P(y) = 0$ are *negative*.

Finally, since $y = z^2$, all the roots of $P(z) = 0$ are pairs of conjugate pure imaginary numbers, as was to be shown.

For n odd, $n = 2m+1$,

$$\begin{aligned} P(z) &= z^{2m+1} + \sum c_1 a_2 z^{2m-1} + \cdots + c_1 a_2 \cdots c_{2m} a_{2m+1} z \\ &= z(z^{2m} + \sum c_1 a_2 z^{2m-2} + \cdots + c_1 a_2 \cdots c_{2m} a_{2m+1}). \end{aligned}$$

Thus $P(z) = 0$ has one root zero and the other roots are pairs of conjugate pure imaginary numbers, by the same argument as above.

It is remarked that *if* $a_j < 0$, $c_j < 0$, $j = 1, 2, \cdots, n$. *Theorem* 2.1 *still holds true*, since the product $c_j a_j$ always occurs in all of the above formulas.

If $a_j < 0$ and $c_j > 0$, or $a_j > 0$ and $c_j < 0$, $j = 1, 2, \cdots, n$, the r_j in (2.3) are positive and negative according as j is odd or even, respectively. Hence all the roots of $P(y) = 0$ are real by the Grommer Theorem. In the continued fraction expansion (2.4) of $Q(y)/P(y)$, the d_j are negative and by Theorem 4.1 of [1], the roots of $P(y) = 0$ are positive. Consequently, the following theorem holds.

THEOREM 2.2. *If* $a_j < 0$ *and* $c_j > 0$, *or if* $a_j > 0$ *and* $c_j < 0$, $j = 1, 2, \cdots, n$, *if* n *is even, the roots of equation* (2.1) *are all real and occur in pairs equal and opposite in sign; if* n *is odd, one root is zero and the remaining roots are all real and occur in pairs equal and opposite in sign.*

3. The equation $B(z) = 0$. In this section the equation

$$(3.1)\qquad B(z) = \begin{vmatrix} z + b_1 & a_2 & 0 & \cdots & 0 & 0 \\ c_1 & z + b_2 & a_3 & \cdots & 0 & 0 \\ 0 & c_2 & z + b_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z + b_{n-1} & a_n \\ 0 & 0 & 0 & \cdots & c_{n-1} & z + b_n \end{vmatrix} = 0$$

is studied.

M. J. Moore [6] proved the following theorem concerning equation (3.1) for $a_j > 0$, $c_j > 0$, by similar determinantal equations.

THEOREM 3.1. *The equation* (3.1), where $a_j > 0$, $c_j > 0$, *or* $a_j < 0$, $c_j < 0$, *and the* b_j *are real constants,* $j = 1, 2, \cdots, n$, *has* n *real, distinct roots.*

Proof. On the removal of constant factors from certain rows and columns of the determinant (3.1) and on multiplication of certain rows and columns by i, $B(iz)$ may be written in the form

$$(3.2)\quad B(iz) = K\cdot\begin{vmatrix} d_1z+k_1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & d_2z+k_2 & -1 & \cdots & 0 & 0 \\ 0 & 1 & d_3z+k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_{n-1}z+k_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & d_nz+k_n \end{vmatrix},$$

where the d_j are *real and positive* and the k_j are *pure imaginary or zero.* Here $B(iz)/K'$ ($K'=K\cdot\prod_{j=1}^{n}d_j$) is a polynomial of the form (1.2) of [1], *i.e.*,

$$P(z)/K'=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n,\ a_j=p_j+iq_j,\ j=1,\ 2,\ \cdots,\ n.$$

If one forms

$$Q(z)/K' = p_1z^{n-1} + iq_2z^{n-2} + p_3z^{n-3} + iq_4z^{n-4} + \cdots \qquad \text{(cf. (1.3) of [1])}$$

in order to set up the expansion (2.4), one finds $Q(z)=0$ since $B(iz)$ is of the form $H_n(z)$ (cf. (4.3) of [1]) and the d_j are real and positive. But then all the roots of $P(z)/K'=0$ lie on the imaginary axis and are distinct ([1], pp. 152–153). Consequently, all the roots of $B(iz)=0$ lie on the imaginary axis and the roots of $B(z)=0$ therefore lie on the real axis.

It may be remarked that Theorem 2.2 is a corollary of Theorem 3.1, where $b_j=0$ and $k_j=0$. It does not, however, include the statement concerning the distribution of the roots on the real axis.

Theorem 2.1 may also be proved in the same way. Here $b_j=0$ and (2.1) may be written in the form $B(z)=0$ with $k_j=0$, so that all roots of (2.1) are pure imaginary or zero.

It may also be remarked that if $a_j<0$, $c_j>0$, or $a_j>0$, $c_j<0$, the determinant (3.1) cannot be transformed into a determinant of the form (3.2), where the c_j are real and positive. Consequently, if $a_j<0$, $c_j>0$, or $a_j>0$, $c_j<0$, equation (3.1) will in general have complex roots.

The following theorem is a direct consequence of Theorem 4.1 of [1].

THEOREM 3.2. *The equation*

$$(3.3)\quad \begin{vmatrix} d_1z+k_1+1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & d_2z+k_2 & -1 & \cdots & 0 & 0 \\ 0 & 1 & d_3z+k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_{n-1}z+k_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & d_nz+k_n \end{vmatrix} = 0,$$

where the d_j are real constants and the k_j are pure imaginary or zero, $j=1,\ 2,\ \cdots,$ n, has k roots with positive real parts and $(n-k)$ roots with negative real parts if k

of the d_j are negative and the remaining $(n-k)$ are positive, provided the determinants D_j (2.2) are different from zero.

There is no loss of generality when one considers the number of roots of equation (3.3) in $R(z)<0$ and $R(z)>0$, since any given half-plane may be transformed into $R(z)<0$ by a linear transformation. Thus, one may formulate this theorem as follows.

THEOREM 3.3. *The equation*

$$\begin{vmatrix} d_1a(z+m)+k_1+1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & d_2a(z+m)+k_2 & -1 & \cdots & 0 & 0 \\ 0 & 1 & d_3a(z+m)+k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_{n-1}a(z+m)+k_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & d_na(z+m)+k_n \end{vmatrix} = 0,$$

where $|a|=1$, $m>0$, the d_j are real constants, and the k_j are pure imaginary, $j=1, 2, \cdots, n$, has k roots in the open half-plane $\bar{a}z+a\bar{z}+2m>0$, and $(n-k)$ roots in the open half-plane $\bar{a}z+a\bar{z}+2m<0$ if k of the d_j are negative and $(n-k)$ of the d_j are positive, provided the determinants D_j (2.2) are different from zero, i.e., provided the expansion (2.4) exists.

An upper bound for the moduli of the roots of equation (3.3) is the maximum modulus of all points on all the Cassinian ovals

$$(3.4)\qquad \begin{aligned} |(d_1z+k_1+1)(d_2z+k_2)| &= \frac{1}{g_1}, \\ |(d_2z+k_2)(d_3z+k_3)| &= \frac{1}{(1-g_1)g_2}, \cdots, \\ |(d_{n-1}z+k_{n-1})(d_nz+k_n)| &= \frac{1}{(1-g_{n-2})g_{n-1}}, \end{aligned}$$

where $0<g_j<1$, $j=1, 2, \cdots, n-1$, (cf. [2], p. 897). The point with largest modulus on each oval is on the line joining its two fixed points.

For example, the determinantal equation

$$(3.5)\quad P(z)=\begin{vmatrix} 2.5z+1-1.25i & -1 & 0 \\ 1 & .059z+.028i & -1 \\ 0 & 1 & -56.874z+55.471i \end{vmatrix} = 0$$

has two roots with negative real parts and one root with positive real part, by Theorem 3.2.

The Cassinian ovals (3.4) for $g_1=56/59$, $g_2=160/168$, are

$$C_1: \quad |(z + .4 - .5i)(z + .5i)| = 50/7,$$
$$C_2: \quad |(z + .5i)(-z + i)| = 25/4.$$

The points of maximum modulus of C_1 and C_2 are $(-1.213, 2.531)$ and $(0, 2.86)$, respectively. Therefore, 2.866 is the maximum modulus of all points of C_1 and C_2, and a very close upper bound to the largest modulus of all the zeros, namely, 2.59.

To obtain a lower bound for the moduli of the roots of (3.5), consider $P'(z) = z^3P(1/z)$. Then the Cassinian ovals (3.4) for $g_1 = .85/1.75$, $g_2 = 1.75/2.04$ are

$$C_3: \quad |(z + .04 - .46i)(-z - 1.7i)| = 1/.85,$$
$$C_4: \quad |(-z - 1.7i)(z - .3i)| = 10/9.$$

The maximum modulus of all points of C_3 and C_4 is 2.15. Hence a lower bound for the moduli of the roots of equation (3.5) is $1/2.15 = .47$.

Computation of the roots of (3.5) by the methods of [2] gives the approximate values as $-.2548 + 2.5780i$, $-.1684 - 2.5754i$, $.0232 + .9974i$.

4. Characteristic equations. By the Grommer Theorem the roots of equation (3.1) are not necessarily real if more elements are non-zero, for example,

$$\begin{vmatrix} z & a_2 & b_2 \\ c_1 & z & a_3 \\ 0 & c_2 & z \end{vmatrix} = 0, \qquad a_j > 0,\ c_j > 0,\ j = 1, 2, 3.$$

Similarly, the roots of equation (2.1) are not necessarily pure imaginary or zero if more elements are non-zero. However, the following special case is in a certain sense an extension of Theorem 2.1 to non-zero elements, but it does limit the value of the elements. This equation is considered, for instance, by Kowalewski ([5], p. 139), who gives the theorem that $D(z) = 0$ can have no real roots different from zero (with no conditions on the a_{jj}).

THEOREM 4.1. *If, in the equation*

$$D(z) = \begin{vmatrix} a_{11} + z & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} + z & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} + z \end{vmatrix} = 0, \tag{4.1}$$

$a_{jk} = -a_{kj}$, $a_{jj} = 0$, and the a_{jk} are real constants, $j, k = 1, 2, \cdots, n$, all the roots of $D(z) = 0$ are pairs of conjugate pure imaginary numbers if n is even, and one root is zero and $(n-1)$ roots are pairs of conjugate pure imaginary numbers if n is odd.

The equation $D(-z) = 0$ (cf. (4.1)) is the *characteristic equation of the matrix* (a_{jk}), $j, k = 1, 2, \cdots, n$, and the negative roots of (4.1) are the *characteristic*

roots of the matrix.

Theorem 4.1 may be proved by the theorems of Grommer and Frank in exactly the same way as Theorem 2.1, since

$$D(z) = z^n + S_2 z^{n-2} + S_4 z^{n-4} + \cdots = 0,$$

where the S_j, $j=2, 4, \cdots$, are the sums of all j-rowed principal minors of the determinant $\|a_{jk}\|$, $j, k=1, 2, \cdots, n$, and are all positive by the properties of such determinants.

In a similar way, Theorem 4.2 may be shown.

THEOREM 4.2. *The equation* $D(z)=0$ (4.1) *has all roots real if the constants* a_{jk} *are real and* $a_{jk}=a_{kj}$, $j, k=1, 2, \cdots, n$.

The following theorems are somewhat more general. The first can be shown directly from the Grommer Theorem. Theorem 4.4 can be proved by an application of the Grommer Theorem to $D(iz)=0$, for, if the roots of $D(iz)=0$ are real, the roots of $D(z)=0$ are pure imaginary or zero.

THEOREM 4.3. *The equation* $D(z)=0$ (4.1) *has all roots real if the constants* a_{jk}, $j, k=1, 2, \cdots, n$, *are real, and if for every pair* j, k *of the* $(n-1)$ *indices* $2, 3, \cdots, n$,

$$a_{1j}a_{jk}a_{k1} = a_{j1}a_{kj}a_{1k}$$

and $a_{1j}a_{j1}$ *is positive* [7].

THEOREM 4.4. *The equation* $D(z)=0$ (4.1) *has all roots pure imaginary or zero if the constants* a_{jk}, $j, k=1, 2, \cdots, n$, *are real,* $a_{jj}=0$, *and if for* $j, k, m=1, 2, \cdots, n$,

$$a_{jk}a_{km}a_{mj} = -a_{kj}a_{mk}a_{jm}$$

and $a_{jk}a_{kj}$ *is negative* [7].

Under the conditions of Theorem 4.4, if $a_{jj}\neq 0$, $j=1, 2, \cdots, n$, the roots of (4.1) will in general be complex.

The following well-known theorem is a further extension of the above, and is easily shown by the continued fraction expansion for $D'(z)/D(z)$ from equation (4.1) and the Grommer Theorem. It has been proved in many other ways (cf., for example, Perron [10], p. 19). Bounds for the roots are also well known ([10], pp. 36 ff.).

THEOREM 4.5. *If* a_{jk} *and* a_{kj} *are conjugate complex constants and the* a_{jj} *are real constants,* $j, k=1, 2, \cdots, n$, *all the roots of equation* (4.1) *are real.*

If one forms the continued fraction expansion for $D'(iz)/D(iz)$ from (4.1), where a_{jk} and $-a_{kj}$ are conjugate complex constants and the a_{jj} are pure imaginary or zero, $j, k=1, 2, \cdots, n$, by the Grommer Theorem it is found that $D(iz)=0$ has all roots real and hence $D(z)=0$ all roots pure imaginary or zero.

Consequently the following theorem holds.

THEOREM 4.6. *The equation* (4.1) *has all roots pure imaginary or zero if* a_{jk} *and* $-a_{kj}$ *are conjugate complex constants and the* a_{jj} *are pure imaginary or zero,* $j, k = 1, 2, \cdots, n$.

Theorem 4.7 (cf. [10], p. 20) is a generalization of the above, and can be proved in the same manner by the continued fraction expansion for $R'(z)/R(z)$ and the Grommer Theorem.

THEOREM 4.7. *The equation*

$$(4.2) \qquad R(z) = \begin{vmatrix} a_{11} + b_{11}z & a_{12} + b_{12}z \cdots a_{1n} + b_{1n}z \\ a_{21} + b_{21}z & a_{22} + b_{22}z \cdots a_{2n} + b_{2n}z \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1} + b_{n1}z & a_{n2} + b_{n2}z \cdots a_{nn} + b_{nn}z \end{vmatrix} = 0$$

has real roots if $\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n} a_{\lambda\mu}z_\lambda w_\mu$, $\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n} b_{\lambda\mu}z_\lambda w_\mu$ *are Hermitian forms* (*i.e.*, $a_{\lambda\mu} = \bar{a}_{\mu\lambda}$, $b_{\lambda\mu} = \bar{b}_{\mu\lambda}$) *and the latter form is positive* (*or negative*) *definite.*

Still further generalizations of this theorem have been developed by M. Parodi for determinantal equations of order n with elements made up of polynomials of degree m [9].

The following theorem can be shown with the use of a theorem on determinants (cf., for instance, Hadamard [4] or Taussky [11]), namely, that the determinant with complex elements $\|a_{jk}\| \neq 0$, $j, k = 1, 2, \cdots, n$, if the elements are such that $|a_{jj}| > A_j$, where A_j denotes the sum of the moduli of the non-diagonal terms of the j-th row of the determinant $\|a_{jk}\|$. However, it can also be readily proved by the above methods, *i.e.*, the continued fraction expansion (2.4) and Theorem 4.1 of [1]. The latter, of course, depends on an expansion of the determinant first. There are a number of techniques for the development of such expansions (cf. Wayland [12]).

THEOREM 4.8. *If the* a_{jk}, $j, k = 1, 2, \cdots, n$, *are complex constants, a sufficient condition that equation* (4.1) *have only roots with negative real parts is that*

$$R(a_{jj}) > \sum_{k \neq j} |a_{jk}|.$$

A sufficient condition that equation (4.1) *have only roots in the lower half of the complex plane is that*

$$I(a_{jj}) > \sum_{k \neq j} |a_{jk}|.$$

The roots lie interior to the circles

$$|a_{jj} + z| = \sum_{k \neq j} |a_{jk}|.$$

Bibliography

1. E. Frank, On the zeros of polynomials with complex coefficients, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 144–158.
2. E. Frank, The location of the zeros of polynomials with complex coefficients, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 890–898.
3. J. Grommer, Ganze transzendente Funktionen mit lauter reellen Nullstellen, J. Reine Angew. Math., vol. 144, 1914, pp. 114–165.
4. J. Hadamard, Leçons sur la Propagation des Ondes, Hermann, Paris, 1903, pp. 13–14.
5. G. Kowalewski, Einführung in die Determinantentheorie, 3d ed., Chelsea, New York, 1948.
6. M. J. Moore, A simple test for the reality and sign of the roots of two determinantal equations of high degree, Math. Gaz., London, vol. 34, 1950, pp. 94–98.
7. T. Muir and W. H. Metzler, A Treatise on the Theory of Determinants, Longmans, Green & Co., New York, 1933.
8. M. Parodi, Sur une application d'un théorème de M. J. Hadamard, Bull. Sci. Math. (2), vol. 72, 1948, pp. 136–138.
9. M. Parodi, Sur quelques proprietés d'une équation algébrique, Bull. Sci. Math. (2), vol. 74, 1950, pp. 121–130.
10. O. Perron, Algebra, 3d ed., vol. 2, Gruyter, Berlin, 1951.
11. O. Taussky, A recurring theorem on determinants, this MONTHLY, vol. 56, 1949, pp. 672–676.
12. H. Wayland, Expansion of determinantal equations into polynomial form, Quart. App. Math., vol. 2, 1945, pp. 277–306.

MATHEMATICAL NOTES

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A NOTE ON THE AUTOMORPHISM GROUP OF A FINITE GROUP

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In this note a theorem is proved about the relationship of the order of a finite group to the order of its automorphism group. The role of this theorem is to exhibit certain prime divisors of the order of the automorphism group. Although the result is natural and easy to prove, we have not been able to locate it anywhere.

Let G be a finite group of order $o(G)$, and let $A(G)$ be the automorphism group of G of order $o[A(G)]$. The theorem we prove is:

THEOREM. *If p is a prime number, and $p^2 \mid o(G)$, then $p \mid o[A(G)]$.*

Since our proof involves a reduction to the case in which G is an Abelian group of prime power order, we first prove the

LEMMA. *If G is an Abelian group of order p^m, $m \geqq 2$, then $p \mid o[A(G)]$.*

Proof. If G is cyclic, $o[A(G)]=(p-1)p^{m-1}$ [1, p. 116], and since $m-1\geqq 1$, $p\mid o[A(G)]$. If, on the other hand, G is of type $(p, p, \cdots, p)$ the lemma is also true since $o[A(G)]=(p^m-1)(p^m-p)(p^m-p^2)\cdots(p^m-p^{m-1})$ [1, p. 112]. In any of the other possibilities for G, G is the direct product of cyclic subgroups $S_1, S_2, \cdots, S_t$ of orders $p^{m_1}, p^{m_2}, \cdots, p^{m_t}$ respectively, where, without loss of generality, we may assume that $t>1$ and that $m_1\geqq 2$. Since S_1 is cyclic of order p^{m_1}, $m_1\geqq 2$, it has an automorphism ϕ of order p. Every $a\in G$ can be written uniquely as $a=s_1s_2\cdots s_t$ where each $s_i\in S_i$. We define ϕ^* by $\phi^*(a)=\phi(s_1)s_2\cdots s_t$. Clearly ϕ^* is an automorphism of G. Moreover ϕ^* is of order p since ϕ is of order p. Thus $p\mid o[A(G)]$.

Now that the lemma has been established we are in a position to prove the aforementioned theorem. Suppose that $p^2\mid o(G)$, and $p\nmid o[A(G)]$. Since $A(G)$ contains as a subgroup the group $I(G)$ of inner automorphisms, and since $I(G)$ is isomorphic to the group G/C, where C is the center of G, then the fact that $p\nmid o[A(G)]$ implies that $p\nmid o(G/C)$; thus S_p, the Sylow subgroup associated with the prime p, must be contained in the center. Hence S_p is a normal subgroup of G. Moreover, since $S_p\subset C$, S_p is in the center of its normalizer, so by a Theorem of Burnside [1, theorem 4, p. 139], G contains a normal subgroup N whose index in G is the order of S_p. It is then clear that $S_p\cap N=1$ and that $S_pN=G$. That is, G is the direct product of S_p and N. Since $p^2\mid o(G)$, S_p is of order p^m with m at least 2. Since $S_p\subset C$, it is Abelian, so by our lemma, S_p possesses an automorphism ϕ of order p. Now every $a\in G$ can be represented uniquely as $a=sn$ where $s\in S_p$, $n\in N$ and where $sn=ns$. We define ϕ^* for G by $\phi^*(a)=\phi(s)n$. Clearly ϕ^* is an automorphism of G, and since ϕ is of order p, ϕ^* is also of order p. ϕ^* being in $A(G)$ then gives us that $p\mid o[A(G)]$, contrary to our assumption that $p\nmid o[A(G)]$. Thus the theorem is proved.

Bibliography

1. Zassenhaus, Hans: Theory of Groups (English Translation), New York, 1949.

DEDUCTION OF CARDANO'S FORMULA BY CONFORMAL MAPPING

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An algebraic function $w=a_0+a_1z+\cdots+a_nz^n$, $a_n\neq 0$, maps the simple z-plane onto an n-sheeted Riemann surface. Inversely, if we must find the values of z corresponding to a given w, the w-surface must be mapped onto the simple z-plane. The problem of solving an algebraic equation of this kind appears therefore related to a problem of conformal mapping.

We will illustrate this principle by solving the equation of third degree

$$z^3-3z-2w=0.$$

We shall get Cardano's formula in a slightly different form, that is to say

$$z = \sqrt[3]{w + \sqrt{w^2 - 1}} + \frac{1}{\sqrt[3]{w + \sqrt{w^2 - 1}}}. \tag{1}$$

It is unimportant which value is given to the square root, but the three values of the cube root will furnish the three values of z. Naturally, if we take a certain value of the square or the cube root in the first summand, the same value must be chosen in the second one. By evaluating the expression z^3-3z, we can verify at once that (1) is a solution.

Since any equation of third degree with simple roots can be transformed by a linear transformation into the equation $z^3-3z-2w=0$, our formula gives the solution for any equation of this kind.

The function $w=\frac{1}{2}(z^3-3z)$ maps the simple z-plane onto a three-sheeted Riemann surface. The equation $w'=\frac{3}{2}(z^2-1)=0$ has the simple roots $z=1$ and $z=-1$. Therefore the points $w(1)=-1$ and $w(-1)=1$ are simple branch-points of the surface. There are no other finite branch-points, but at $w=\infty$ there is a double branch-point. The only value of z for which $w(z)=\infty$ is $z=\infty$.

If we succeed in mapping the w-surface onto the z-plane in such a way that the simple points $w=\pm 1$ are transformed into the double points $z=\mp 1$, the simple point $w=\infty$ into the triple point $z=\infty$, and all other simple w-points conformally into simple z-points, then we will have solved the equation $z^3-3z-2w=0$.

But we will not begin with the mapping of the w-surface mentioned above, this surface not being simple enough. It is easier first to map a six-sheeted surface of the variable, say v, which is regularly branched,* onto the simple z-plane. At a later stage in the argument we will return to the three-sheeted w-surface. The surface of the variable v shall have the following properties: (a) it is six-sheeted and simply connected, (b) its only branch-points are at the points $v=-1$, $v=1$ and $v=\infty$, (c) at each of the points $v=-1$ and $v=1$ are three simple branch-points and at $v=\infty$ are two double branch-points. This surface will therefore be regularly branched.

Cutting the surface of the variable v along the real axis, we get 12 half-planes which we consider as curvilinear triangles with the vertices $v=-1$, $v=1$ and $v=\infty$ and with the angles π, π and π. Now we have only to map one of these triangles onto a certain curvilinear triangle with the angles $\pi/2$, $\pi/2$ and $\pi/3$ respectively, and then by reflection the whole v-surface will be mapped onto a simple v_5-plane. We shall map a lower half-plane of the v-surface onto the following triangle of the v_5-plane: two of its sides are parts of the positive real and positive imaginary axis, meeting at $v_5=0$. This is therefore one of the vertices of the triangle with an angle of $\pi/2$. We close this triangle by an arc of the circle

* A Riemann surface is called regularly branched, if for each fixed base point, every point of the surface lying over this base point is a branch-point of the same order; cf. Nevanlinna, R., Eindeutige analytische Funktionen, Grundlehren, vol. XLVI, Berlin, Springer 1936/Edwards, Ann Arbor, 1944, pp. 270–300. The present note was suggested by material in this book.

which has its center at $v_5=-i/\sqrt{3}$ and passes through $v_5=1$. This triangle, which lies entirely in the first quadrant, has its two other vertices at $v_5=i/\sqrt{3}$ and $v_5=1$ and the corresponding angles are $\pi/2$ and $\pi/3$ (Fig. 1).

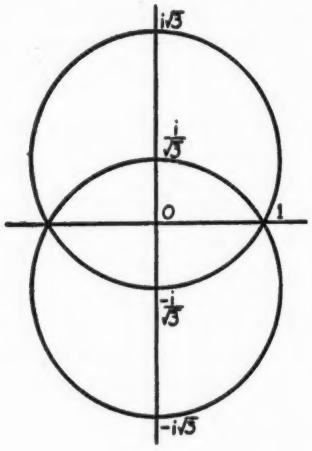

FIG. 1

It is easy to see that the transformations

$$v_1=\frac{1-v}{1+v}, \qquad v_2=\sqrt{v_1}, \qquad v_3=\frac{1+iv_2}{1-iv_2},$$

$$v_4=-\epsilon\sqrt[3]{v_3}, \qquad v_5=\frac{\epsilon-v_4}{\epsilon+v_4}, \quad \text{where} \quad \epsilon=\frac{1}{2}+\frac{i}{2}\sqrt{3},$$

carried out one after the other, finally map the half-plane $\Im(v)<0$ onto the triangle in question. Here it is presumed that we take $\sqrt{-1}=+i$ and $\sqrt[3]{-1}=-1$.

Now we put $v_6=v_5^2$. By this transformation, each pair of the v_5-triangles, symmetrical with respect to the origin, goes into a single v_6-triangle. The triangle in the v_5-plane which we discussed before, is now transformed into the following one: the segments of the real axis $-\frac{1}{3}\leqq v_6\leqq 0$ and $0\leqq v_6\leqq 1$ are two of its sides, the origin is one of the vertices with π as the corresponding angle. The third side is formed by an analytic curve, cutting the real axis in $-\frac{1}{3}$ and 1 at angles of $\pi/2$ and $2\pi/3$ respectively and lying in the upper half-plane (Fig. 2).

Between the v-surface and the v_5-plane there existed a 1-1 correspondence; more precisely, there existed a 1-1 correspondence between every one of the triangles in the v_5-plane (Fig. 1) and one of the half-planes of the v-surface.

In general there lie exactly six points over each base point of the v-surface, which map in a 1-1 manner onto the points of the v_5 plane. These six points can be grouped into three pairs in such a way that for each pair the corresponding

value of v_6 is the same. Thus there is a 1-1 mapping of the v_6-plane onto the Riemann surface which is obtained by identifying in the primitive v-surface the two half-planes which map into the same v_6 triangle.

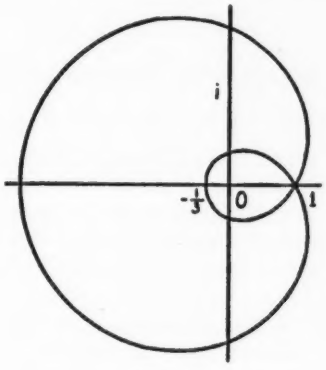

FIG. 2

After this identification is performed, the v-surface will have only three sheets. As both of the triple points $v_5=-1$ and $v_5=1$ are now transformed into the point $v_6=1$, this three-sheeted surface will have only one double branch-point, at $v=\infty$. Before that we had three image-points of $v=-1$ in the v_5-plane, *i.e.* the points $v_5=0$ and $v_5=\pm i\sqrt{3}$, each of which was to be counted as a double point. By $v_6=v_5^2$, the points $v_5=-i\sqrt{3}$ and $v_5=i\sqrt{3}$ go into the same point $v_6=-3$, which must also be counted as a double point, whereas the point $v_5=0$ remains at $v_6=0$, but changes from a double point into a simple point. Therefore, the new three-sheeted surface will have a simple branch-point and a simple sheet at the point $v=-1$. We can make the same observations about the image-points of $v=+1$, namely, there is a double point at $v_6=-\frac{1}{3}$ and a simple point at $v_6=\infty$.

The three-sheeted surface has therefore the following branch-points: a double one at ∞, and at $+1$ as well as at -1 there is a simple branch-point and a simple sheet. This surface is therefore identical with the surface of the variable w mentioned at the beginning.

Now we have only to use a linear transformation to send the point $v_6=1$ into $v_7=\infty$, the point $v_6=-\frac{1}{3}$ into $v_7=-1$, and the point $v_6=-3$ into $v_7=1$. This is done by the transformation $v_7=2(v_6+1)/(v_6-1)$. The three-sheeted w-surface is now mapped onto a simple plane, the image-point of its double branch-point is at $v_7=\infty$, and the image-points of its simple branch-points are at $v_7=1$ and $v_7=-1$. Therefore we can put $v_7=z$.

The transformations $v_1(v)$, $v_2(v_1)$, $v_3(v_2)$ and $v_4(v_3)$ yield $v_4=-\epsilon\sqrt[3]{v+\sqrt{v^2-1}}$, and the transformations $v_5(v_4)$, $v_6(v_5)$ and $v_7(v_6)$ yield $v_7=\epsilon^2 v_4+(1/\epsilon^2 v_4)$. So we have the final formula

$$z = \sqrt[3]{w + \sqrt{w^2 - 1}} + \frac{1}{\sqrt[3]{w + \sqrt{w^2 - 1}}}$$

as the solution of the equation

$$z^3 - 3z - 2w = 0.$$

A CERTAIN CUBIC TRANSFORMATION

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Take a fixed point $A_3(0, 0, 1)$ as origin, and take as the fixed fundamental base curve a cubic of the syzygetic pencil, $F=x_1^3+x_2^3+x_3^3-3\lambda x_1x_2x_3=0$. Points collinear with the origin and conjugate with respect to the base are said to be inverse. A line through the origin and a generic point $P'(x_1', x_2', x_3')$ will intersect the polar line of P' with respect to the cubic in a point $P(x_1, x_2, x_3)$. P and P' are conjugate points.

The only case considered here is the one in which A_3P' and the polar line of P' with respect to the base cubic F are distinct.

The polar line of P' with respect to F is also the polar line of the first polar of P' with respect to F, that is, the polar line of the polar conic of P' with respect to F. Thus to determine the cubic inverse of a point P', determine the polar conic of P', and use the ordinary construction for pole and polar with respect to the conic.

When the point $P(x_1, x_2, x_3)$, the intersection of A_3P' and the polar line of P', is found, a relation is established between x_1', x_2', x_3' and x_1, x_2, x_3 by the equations $x_1:x_2:x_3=f_1:f_2:f_3$ where $f_1=\lambda x_1'^2x_2'-x_1'x_3'^2$, $f_2=\lambda x_1'x_2'^2-x_2'x_3'^2$, $f_3=x_1'^3+x_2'^3-2\lambda x_1'x_2'x_3'$, and thus there is determined an interesting cubic transformation. Since f_1, f_2, f_3 are rational cubic functions of x_1', x_2', x_3', the transformation is rational. Under this transformation, to a given point $P'(x_1', x_2', x_3')$ corresponds one point $P(x_1, x_2, x_3)$, but to a given point P will correspond not one point P' but a finite number, $n>1$. The points P' corresponding to a given point P are determined by the intersections of the three curves whose equations are

$$\frac{f_1}{x_1} = \frac{f_2}{x_2} = \frac{f_3}{x_3}$$

or

(a) $x_2f_1 - x_1f_2 = 0$; (b) $x_3f_1 - x_1f_3 = 0$; (c) $x_3f_2 - x_2f_3 = 0$.

These curves have nine points of intersection since equation (a) is a linear combination of (b) and (c). Thus if $f_1=0$, $f_2=0$, $f_3=0$ have no common point of intersection, to a given point P will correspond nine points P', but if $f_1=0$, $f_2=0$, $f_3=0$ do have one common point of intersection, then since the position of this point is independent of the position of P, it may be disregarded, and to a point P will correspond only eight points P', *etc.* It can be proved that $f_1=0$,

$f_2=0, f_3=0$ intersect in the following seven points:

(1) $(0, 0, 1)$

(2) $[\lambda, (\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{2/3}, \lambda(\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{1/3}]$

(3) $[\lambda, (\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{2/3}, \lambda(\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{1/3}]$

(4) $[\lambda, (\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega^2, \lambda(\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega]$

(5) $[\lambda, (\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega, \lambda(\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega^2]$

(6) $[\lambda, (\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega^2, \lambda(\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega]$

(7) $[\lambda, (\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega, \lambda(\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega^2]$

where ω and ω^2 are cube roots of unity. Thus to each given point P will correspond two points P', and this cubic transformation sets up a (1, 2) correspondence.

The equations of the transformation which determines the two points P' corresponding to a given point P can be obtained by using equations (a), (b), (c). The equations of the transformation are:

$$x_1' : x_2' : x_3' = x_1(\lambda x_1x_2 \pm \sqrt{\lambda^2x_1^2x_2^2 - x_3(x_1^3 + x_2^3 - \lambda x_1x_2x_3)})$$
$$: x_2(\lambda x_1x_2 \pm \sqrt{\lambda^2x_1^2x_2^2 - x_3(x_1^3 + x_2^3 - \lambda x_1x_2x_3)}) : x_1^3 + x_2^3 - \lambda x_1x_2x_3.$$

Some very interesting results are obtained when λ is assigned any one of the cube roots of unity as its value. Under these conditions consider again the seven points (1), (2), (3), (4), (5), (6), (7). In all three cases (1) remains the same, (2) coincides with (3), (4) with (6), and (5) with (7). These coincidences determine the following nine points:

		Points (2), (3)	Points (4), (6)	Points (5), (7)
(S):	$\lambda = \omega$,	I$(1, \omega^2, 1)$,	IV$(1, \omega, \omega)$,	VII$(1, 1, \omega^2)$,
(R):	$\lambda = 1$,	II$(1, 1, 1)$,	V$(1, \omega^2, \omega)$,	VIII$(1, \omega, \omega^2)$,
(T):	$\lambda = \omega^2$,	III$(1, \omega, 1)$,	VI$(1, 1, \omega)$,	IX$(1, \omega^2, \omega^2)$.

These nine points lie by threes on each of nine lines. The points that lie in the same column are collinear, and if a point is chosen from each row and each column, those three points are collinear. The nine lines pass by threes through the vertices of the fundamental triangle.

In the original syzygetic pencil of cubics, the nine common points of intersection of all the cubics are their inflexion points. These nine inflexion points lie by threes on twelve lines—the Maclaurin lines. The Maclaurin lines intersect in twelve other points called the "critic centers" of the cubic F. The nine points determined by the coincidences when $\lambda=1$, ω, or ω^2, and the point (1) are ten of these critic centers.

A GENERAL RULE FOR DIVISIBILITY BASED ON THE DECIMAL EXPANSION OF THE RECIPROCAL OF THE DIVISOR

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1. The following rule provides a general test for determining the divisibility of an integer $N=\sum_{i=0}^{k} a_i 10^i$ by any integer n:

1. Find p and q, where p is the number of terms preceding the period in the decimal expansion of $1/n$ and q is the number of terms in the period. They can be found directly or by applying the theorem which states that, if $n=n_0 2^\alpha 5^\beta$ where n_0 and 10 are relatively prime, then $p=\max(\alpha, \beta)$ and q is the exponent to which 10 belongs modulo n_0. (The number of digits in N must exceed $p+q$ for the rule to have any significance.)

2. Mark off the digits of N in groups of q beginning with the $(p-1)$th digit from the right. (Up to $q-1$ zeros may have to be placed before N to fill out the last group; this gives $[(k-p)/q]+1$ groups.)

3. Find the sum, S, of the q-digit numbers formed in 2.

4. Substitute S for the digits of N preceding the last p digits, obtaining the smaller number N'.

The rule states that if N' is divisible by n, so is N.

Proof. Since

$$N = \sum_{i=0}^{p-1} a_i 10^i + \sum_{j=0}^{[(k-p)/q]} \sum_{i=0}^{q-1} a_{p+jq+i} 10^{p+jq+i}$$

and

$$N' = \sum_{i=0}^{p-1} a_i 10^i + \sum_{j=0}^{[(k-p)/q]} \sum_{i=0}^{q-1} a_{p+jq+i} 10^{p+i},$$

$N-N'$ consists of a series of terms of the form

$$a_{p+jq+i}(10^{p+jq+i} - 10^{p+i}).$$

Since, in the decimal expansion of $1/n$, the remainder after $p+jq+i$ terms and after $p+i$ terms are equal,

$$10^{p+jq+i} \equiv 10^{p+i} \pmod{n}$$

and, therefore,

$$N \equiv N' \pmod{n}.$$

2. A simpler rule can be stated if a remainder of $n-1$ appears for the first time after r terms have been found in the decimal expansion of $1/n$. When this happens, $p=0$ and $q=2r$. Alternatively, if n is prime, and if $1/n$, which is purely periodic, has a period $2r$, the remainder after r terms is $n-1$. In such cases, the rule is to mark off the digits of N in groups of r beginning with the last digit, and to obtain N'' as the difference between the sums of the odd and even

groups, respectively, of the r-digit numbers. (The test has meaning only if the number of digits of N exceeds r). If N'' is divisible by n, so is N.

Proof. The truth of the first statement is seen from the fact that, if the remainder after r terms is $n-1$, the remainder after $2r$ terms is found by dividing $(n-1)10^r$ by n. But, since

$$10^r \equiv n - 1 \pmod{n},$$
$$(n - 1)10^r \equiv (n - 1)^2 \equiv 1 \pmod{n}.$$

A remainder of 1 implies that $p=0$. Also, since a remainder of 1 could not have occurred before the $2r$-th term, $q=2r$. The truth of the second statement can be shown as follows: let the remainder after r terms be l, where $1<l<n$. Then

$$10^r \equiv l \pmod{n}$$

and

$$l10^r \equiv l^2 \pmod{n}.$$

But

$$l10^r \equiv 1 \pmod{n},$$

so that

$$l^2 \equiv 1 \pmod{n}.$$

Since n is prime and $l=1$ is excluded as a solution, this has only one solution $n-1$.

The rule can now be proved by writing:

$$N = \sum_{j=0}^{[k/r]} \sum_{i=0}^{r-1} a_{jr+i}10^{jr+i}$$

and

$$N'' = \sum_{j=0}^{[k/r]} \sum_{i=0}^{r-1} (-1)^j a_{jr+i}10^i.$$

$N-N''$ therefore consists of a series of terms of the form

$$a_{jr+i}(10^{jr+i} - (-1)^j 10^i).$$

But, since

$$10^r \equiv (n - 1) \pmod{n}$$
$$10^r \equiv -1 \pmod{n}$$
$$10^{jr} \equiv (-1)^j \pmod{n}$$
$$10^{jr+i} \equiv (-1)^j 10^i \pmod{n}$$

it follows that

$$N \equiv N'' \pmod{n}.$$

Examples:

For $n=9$:

$$p = 0,\ q = 1,\ \text{and}\ N' = a_0 + a_1 + a_2 + \cdots .$$

For $n=11$:

$$p = 0,\ q = 2,\ \text{and}\ N' = (a_0 + 10a_1) + (a_2 + 10a_3) + \cdots$$

or

$$r = 1 \text{ and } N'' = a_0 - a_1 + a_2 - \cdots .$$

For $n=10^u+1$:

$$r = u \text{ and } N'' = (a_0 + 10a_1 + \cdots + 10^{u-1}a_{u-1}) - (a_u + 10a_{u+1} + \cdots + 10^{u-1}a_{2u-1}) + \cdots .$$

For $n=7$ or 13:

$$r = 3 \text{ and } N'' = (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + \cdots .$$

The usual rule for 7,

$$N''' = (a_0 + 3a_1 + 2a_2) - (a_3 + 3a_4 + 2a_5) + \cdots$$

can easily be derived from this.

For $n=12$:

$$p = 2,\ q = 1 \text{ and } N' = a_0 + 10a_1 + 100(a_2 + a_3 + \cdots).$$

This can be reduced to

$$N''' = a_0 - 2a_1 + 4(a_2 + a_3 + \cdots).$$

For $n=2^\alpha 5^\beta$:

$$p = \max(\alpha, \beta),\ q = 0,\ \text{and}\ N' = a_0 + 10a_1 + \cdots + 10^{p-1}a_{p-1}.$$

CLASSROOM NOTES

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All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE PROOF OF THE LAW OF SINES

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Most mistakes made by mathematicians consist in neglecting to consider a certain possibility. A proof or a theorem can become invalid if two roots of an equation become equal, if a determinant becomes zero, if the factorization in the considered number field is not unique, and so on. Often it is possible to complete the proof for the exceptional case, but sometimes the theorem is wrong. It is therefore of importance that the student learn as early as possible to check if a proof is really complete.

It is surprising that in a large number of textbooks on trigonometry the proof for the law of sines is incomplete.

In general, the proof is given by drawing one of the altitudes. Then two cases are considered, namely the case that each angle is less than 90° and the case that one of the angles is greater than 90°. Two figures illustrate the proof.

The authors of these books seem to believe that it is not necessary to prove the theorem for the case that one of the angles equals 90° since the trigonometric functions can directly be used for right triangles. But this is not correct.

Given two sides a and b, and the angle B opposite to the smaller side b. In this case we are not permitted to use the law of sines if this theorem is not proved for right triangles since it can be that the angle $A=90°$. But we cannot use the trigonometric functions either, since we do not know that the triangle has a right angle.

In these books the law of sines is used and it is explicitly stated that A can be equal to 90° and that we have one and only one solution in this case. But it is not recognized that here a theorem is used in a case in which it is not proved.

If a, b, and B are given, but not numerically, we cannot determine whether the triangle is a right one or not. If the law of sines is not proved for right triangles, a complete solution must consist of two parts, namely assuming that no angle equals 90°, and assuming that one angle equals 90°. If a, b, and B are given numerically, it can happen that $\sin B = b/a$ if a 4-place table is used, but $\sin B \neq b/a$ if a 5-place table is used, and the method of solution would depend on the kind of table to be used.

This shows that it is necessary to prove the law of sines for right triangles. Of course, using continuity it is obvious that the theorem is correct for right triangles, too. But also the elementary proof is so simple that there is no reason to omit the proof of this case in any book or class.

CAUCHY'S FORM OF R_n FROM THE ITERATED INTEGRAL FORM

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Professor C. P. Nicholas might have remarked, in his happy article, *Taylor's Theorem in a first course* (this MONTHLY, vol. 58, no. 8, October, 1951, pp. 559–562), that Cauchy's form of the remainder, as well as Lagrange's, may be rather easily deduced from the iterated integral form. We need only to apply the geometrically obvious equality

$$\int_a^b \int_a^s \phi(t, s)\,dt\,ds = \int_a^b \int_t^b \phi(t, s)\,ds\,dt$$

repeatedly, "from the inside out," as indicated:

$$\begin{aligned}
R_n &= \int_a^x \int_a^y \cdots \int_a^z \int_a^u \underbrace{\int_a^v \int_a^w} f^{(n)}(t)\,dt\,dw\,dv\,du \cdots dy \\
&= \int_a^x \int_a^y \cdots \int_a^z \underbrace{\int_a^u \overbrace{\int_a^v \int_t^v}} f^{(n)}(t)\,\overbrace{dw\,dt}\,dv\,du \cdots dy \\
&= \int_a^x \int_a^y \cdots \underbrace{\int_a^z \overbrace{\int_a^u \int_t^u}} \int_t^v f^{(n)}(t)\,dw\,\overbrace{dv\,dt}\,du \cdots dy \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \overbrace{\int_a^x \int_t^x} \cdots \int_t^z \int_t^u \int_t^v f^{(n)}(t)\,dw\,dv\,du \cdots \overbrace{dy\,dt} \\
&= \int_a^x f^{(n)}(t) \left\{ \int_t^x \cdots \int_t^z \int_t^u \int_t^v dw\,dv\,du \cdots dy \right\} dt \\
&= \int_a^x f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,dt = f^{(n)}(\xi)\,\frac{(x-\xi)^{n-1}}{(n-1)!}\,(x-a), \quad a \leqq \xi \leqq x,
\end{aligned}$$

which is the desired form. (The last step, which is an application of the simplest mean value theorem for integrals, is to be found in numerous textbooks, the integral which appears in it being a standard formula for R_n.)

A MINUTE THEORY OF RADICAL EQUATIONS

G. B. HUFF and D. F. BARROW, University of Georgia

A section on radical equations is usually found in freshman algebra texts in the chapter on quadratics. There is ordinarily no attempt to develop any theory for radical equations, although there is an admonition to test for "extraneous roots." (See this MONTHLY, vol. 55, p. 639, for an interesting comment on this latter notion.) The texts do not invariably emphasize that the interest in the problem arises from the fact that in these equations $\sqrt{R}$ is defined only for

R real and non-negative and represents the non-negative square root of R. Doubtless the average freshman does not notice the lack of a theory here, but the instructor may, especially when he tries to make up nice problems. In this note we wish to point out two things. Some theory may be obtained, as in the case of quadratic equations, by applying the methods of the book to literal problems. An intuitive geometric idea of what is going on may be obtained from the geometry of the parabola. In both cases the results seem simple enough to be interesting and are certainly useful in designing problems.

We will restrict the strictly algebraic attack to the radical equation

$$\sqrt{Ax+B}+\sqrt{ax+b}=c, \qquad A-a>0, \qquad c>0, \tag{1}$$

where A, B, a, b, and c are in some field K of real numbers. Due to the definition of the symbol $\sqrt{R}$, we seek real numbers, not necessarily in K, which are roots of (1). Let D be defined by:

$$D=(A-a)(Ab-aB)+Aac^2. \tag{2}$$

We shall prove the following:

THEOREM. *The equation* (1) *has no solution if* $D<0$. *If* $D\geqq 0$, *the equation* (1) *has two, one, or no solutions according as two, one, or none of the conditions* (*p*) *and* (*n*) *hold*:

$$\text{(p)}\quad Ac\geqq-\sqrt{D}\geqq ac; \qquad \text{(n)}\quad Ac\geqq\sqrt{D}\geqq ac.$$

By the usual device of eliminating radicals by squaring, we construct the *auxiliary equation*,

$$(A-a)^2z^2+2[(A-a)(B-b)-(A+a)c^2]z + [(B-b)^2-2(B+b)c^2+c^4]=0, \tag{3}$$

and compute its discriminant to be $16c^2D$, where D is given by (2). Since any root of (1) must by agreement be real and must also be a root of (3), then $D<0$ implies that (1) has no solutions of the sort we want.

If $D\geqq 0$, then the auxiliary equation (3) has the root

$$z_+=[(A+a)c^2-(A-a)(B-b)+2c\sqrt{D}\,]/(A-a)^2,$$

and a root z_- which is the same except for the sign on the term $2c\sqrt{D}$. As a first step in checking these as possible roots of (1), we compute that

$$Az_++B=(Ac+\sqrt{D})^2/(A-a)^2; \qquad az_++b=(ac+\sqrt{D})^2/(A-a)^2. \tag{4}$$

Similar formulas hold for Az_-+B and az_-+b. This gives a result that every freshman algebra teacher has checked countless times, namely:

If D *is non-negative and* z_0 *is a root of the auxiliary equation* (3), *then* Az_0+B *and* az_0+b *are non-negative and are in fact squares in* $K(\sqrt{D})$, *the field obtained*

by adjoining $\sqrt{D}$ *to* K.

If $K(\sqrt{D})$ is the field needed to solve the auxiliary equation, then the square roots found in checking are also in $K(\sqrt{D})$.

We now continue with the proof of the theorem by showing that z_+ is a root of (1) if and only if the condition (p) of the theorem holds. It is convenient here to write (p) in the form:

$$(p_1) \quad Ac + \sqrt{D} \geqq 0; \qquad (p_2) \quad -ac - \sqrt{D} \geqq 0.$$

It is clear from the formulas (4) and the convention in equation (1) that $A-a>0$, that (p_1) and (p_2) are sufficient to insure that z_+ is a root of (1). To prove that these conditions are also necessary, we show that if either one or both fail, then z_+ does not satisfy (1).

Suppose that neither (p_1) nor (p_2) is true. Then $-Ac-\sqrt{D}>0$ and $ac+\sqrt{D}>0$ and substitution of z_+ in (1) gives the false statement

$$(-Ac - \sqrt{D})/(A - a) + (ac + \sqrt{D})/(A - a) = c.$$

The argument is a little different if (p_1) holds and (p_2) fails. Now we have $Ac+\sqrt{D}\geqq 0$ and $ac+\sqrt{D}>0$ and substitution yields $(Ac+\sqrt{D})/(A-a)+(ac+\sqrt{D})/(A-a)=c$. This reduces to $ac+\sqrt{D}=0$, the desired contradiction. It follows, in the same way, that z_+ is not a root of (1) when (p_1) fails and (p_2) holds.

Similarly, z_- is a root of (1) if and only if the condition (n) holds. For $D>0$, this completes the proof of the second assertion of the theorem. When $D=0$, $z_+=z_-$, the inequalities (p) and (n) are the same, and equation (1) has one or no roots as this one relation is or is not satisfied.

Another method of discussing the roots of (1) may be found by means of analytic geometry. We shall deal only with the case where there are genuinely two radicals in equation (1), that is, both radicals actually contain x, and the ratio of the radicands is not independent of x. This means that we assume $Aa(Ab-aB)\neq 0$. We begin by plotting in the u, v-plane the graphs of

$$\text{(d)} \quad \sqrt{u} + \sqrt{v} = c, \qquad \text{(e)} \quad \sqrt{u} - \sqrt{v} = c, \qquad \text{(f)} \quad -\sqrt{u} + \sqrt{v} = c; \quad c > 0.$$

The graphs of these three equations are three arcs which form one complete parabola lying wholly in the first quadrant. The vertex of this parabola is the point $(c^2/4, c^2/4)$ and it is tangent to the axes at $(c^2, 0)$ and $(0, c^2)$. The graph of (d) extends from $(0, c^2)$ to $(c^2, 0)$ and contains the vertex. The graph of (e) runs from $(c^2, 0)$ to "infinity" and the graph of (f) runs from $(0, c^2)$ to "infinity." If a tangent line be drawn to the parabola at any point, the sum of the intercepts of this tangent line is c^2. Next, we consider in the u, v-plane the straight line whose parametric equations are

$$\text{(g)} \qquad u = Ax + B, \qquad v = ax + b,$$

where x is the parameter. If this line cuts the graph of (d), the value of the parameter x determined by this point will be a solution of (1). Under our assumption $Aa(Ab-aB)\neq 0$, the line (g) has two non-zero intercepts given by $(aB-Ab)/a$ and $(Ab-aB)/A$. Thus when equation (1) is given, these intercepts are easily computed and an inspection of the figure yields the following criteria.

Case I. *When both intercepts are positive and not greater than c^2, the equation* (1) *has two solutions if the sum of the intercepts is greater than c^2, just one solution if the sum of the intercepts is equal to c^2, and none if the sum of the intercepts is less than c^2.*

Case II. *When only one of the intercepts satisfies the condition of being positive and not greater than c^2, equation* (1) *has exactly one solution.*

Case III. *When neither intercept satisfies the condition of being positive and not greater than c^2, equation* (1) *has no solutions.*

It is clear that both methods give equally simple statements for equations like (1) with different arrangements of signs. With slight changes, both ideas apply to the radical equation $\sqrt{Ax+B}=cx+d$. We close by pointing out exactly how the above results may be used to construct examples to order. Suppose we want a radical equation whose auxiliary equation has rational roots of which only z_- is a root of (1). Having chosen, more or less at random $A=3$, $a=2$, and $c=2$, it is clear that $\sqrt{D}=5$ is a convenient solution of (n) which does not satisfy (p). Substitution of these values in (2) gives $3b-2B=1$. Any solution of this equation gives a problem of the desired sort. The particular solution $B=4$, $b=3$ yields $\sqrt{3x+4}+\sqrt{2x+3}=2$. If we want an auxiliary equation with irrational roots, a slightly different example arises by choosing $\sqrt{D}=\sqrt{23}$ for the same choice of A, a, and c. The condition on B and b is now $3b-2B=-1$, with $B=2$, $b=1$ as a simple solution. The reader may find it interesting to verify that z_- is indeed the only root of $\sqrt{3x+2}+\sqrt{2x+1}=2$. Both of these come under Case II of the geometric criteria.

TERM-WISE DIFFERENTIATION OF POWER SERIES

T. M. Apostol, California Institute of Technology

Let the function $f(x)$ be defined by a power series

$$f(x)=\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

in the convergence-interval $|x|<r$. The purpose of this note is to present an elementary proof of the fact that the derivative $f'(x)$ exists at every point in $|x|<r$ and that it may be obtained by differentiating the series in (1) term-by-term. The usual proofs of this fact make use of the concept of uniform convergence or else involve rearrangement of double series, and consequently

are not given in most beginning calculus texts. Hardy's *Pure Mathematics* does not treat differentiation or integration of power series since he avoids "the inclusion of anything that involves really difficult ideas." The proof given below makes use only of the mean-value theorem of differential calculus and it is felt that the proof is simple enough to be presented to students in sophomore calculus.

It is assumed that the existence of an interval of convergence has been established for power series and that it is known that power series converge absolutely inside the interval. We first prove:

THEOREM 1. *Let the series in equation* (1) *have the interval of convergence* $|x| < r$. *Then the series*

$$\sum_{n=0}^{\infty} n a_n x^{n-1} \tag{2}$$

has the same interval of convergence.

Proof. Let us first take a positive value of x, $0 < x < r$, and let h be a small positive number such as $0 < x < x+h < r$. Then the series for $f(x)$ and for $f(x+h)$ are each absolutely convergent and hence so is the series

$$\frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h}. \tag{3}$$

Applying the mean-value theorem to each term of this series we have

$$\frac{(x+h)^n - x^n}{h} = n\xi_n^{n-1}$$

where $x < \xi_n < x+h$. Hence the series in (3) is identical to the series

$$\sum_{n=0}^{\infty} n a_n \cdot \xi_n^{n-1}, \tag{4}$$

which must be absolutely convergent, since the series in (3) is. The series in (4) is no longer a power series but it dominates the power series (2) so that the series (2) must be absolutely convergent for this x. Hence the interval of convergence of (2) includes the interval $|x| < r$, and it cannot extend beyond this interval since (2) dominates (1). This proves the theorem.

If we apply the theorem to series (2) we obtain the following:

COROLLARY. *The series*

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

has the same interval of convergence as the series in (1).

The next theorem can be obtained as a consequence of Theorem 3 but a direct proof is simple and it also serves as an introduction to the proof of Theorem 3.

THEOREM 2. *The function defined by* (1) *is continuous for each x inside the interval of convergence.*

Proof. Let $0 \leqq |x| < r$ and choose x_1 so that $0 \leqq |x| < |x_1| < r$. Next choose h so that $0 \leqq |x+h| < |x_1| < r$. We will prove that

$$\lim_{h \to 0} |f(x+h) - f(x)| = 0.$$

Proceeding as in the proof of Theorem 1, we have

$$f(x+h) - f(x) = h \sum_{n=0}^{\infty} n a_n \xi_n^{n-1}, \tag{5}$$

where each number ξ_n lies between x and $x+h$. By Theorem 1, the series

$$\sum_{n=0}^{\infty} n a_n x_1^{n-1}$$

is absolutely convergent and since $|\xi_n| < |x_1|$, equation (5) leads to the inequality

$$0 \leqq |f(x+h) - f(x)| \leqq |h| \sum_{n=0}^{\infty} n |a_n| |x_1|^{n-1}. \tag{6}$$

The series multiplying $|h|$ in (6) represents a number independent of h so that if we now let $h \to 0$ the theorem follows.

THEOREM 3. *If the function $f(x)$ is given by the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

in the interval $|x| < r$, then for any x inside this interval the derivative $f'(x)$ exists and is given by the power series

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Proof. Take x and then x_1 so that $0 \leqq |x| < |x_1| < r$. Let

$$F(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

This series converges absolutely in $|x| < r$ by Theorem 1, and our aim is to prove that $f'(x)$ exists and equals $F(x)$ in this interval. Proceeding as in the previous theorem we choose h so that $|x+h| < |x_1| < r$ and obtain

$$F(x) - \frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} n a_n (x^{n-1} - \xi_n^{n-1}), \tag{7}$$

where each number ξ_n lies between x and $x+h$. Now we apply the mean-value theorem again to obtain

$$x^{n-1} - \xi_n^{n-1} = (n-1)\eta_n^{n-2}(x - \xi_n)$$

where each number η_n lies between x and ξ_n. Hence we have

$$F(x) - \frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} n(n-1) a_n \eta_n^{n-2}(x - \xi_n), \tag{8}$$

the series on the right of (8) being absolutely convergent since it is identical to the one in (7). But we have $|x - \xi_n| < |h|$ and $|\eta_n| < |x_1|$ so that we have

$$0 \leqq \left| F(x) - \frac{f(x+h) - f(x)}{h} \right| \leqq |h| \sum_{n=0}^{\infty} n(n-1) |a_n| |x_1|^{n-2}, \tag{9}$$

the series in (9) being convergent by the corollary to Theorem 1. Letting $h \to 0$ in (9) we see that $f'(x)$ exists and equals $F(x)$.

For the sake of completeness we include

THEOREM 4. *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

have the interval of convergence $|x| < r$. *Let*

$$G(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Then the series for $G(x)$ *has the same interval of convergence and*

$$G'(x) = f(x).$$

Proof. Since the power series for $G(x)$ has *some* interval of convergence, Theorem 3 tells us that it must be the same as the interval for $f(x)$ and that $G'(x) = f(x)$.

A METHOD FOR THE SOLUTION OF THE GENERAL CUBIC

T. D. NAGLE, University of Bridgeport

Given the general cubic

$$f(x) = x^3 + bx^2 + cx + d, \qquad d \neq 0. \tag{1}$$

By the transformation $x = 1/y$, we replace the equation $f(x) = 0$ by the equation $g(y) = 0$ where

$$g(y) = \frac{y^3}{d} f\left(\frac{1}{y}\right) = y^3 + \frac{c}{d} y^2 + \frac{b}{d} y + \frac{1}{d}.$$

Case 1. Suppose

$$c^2 - 3bd = 0. \tag{2}$$

Then the linear transformation

$$y = z - \frac{c}{3d}$$

yields

$$g\left(z - \frac{c}{3d}\right) = z^3 + g\left(-\frac{c}{3d}\right),$$

and x_1 is a root of $f(x) = 0$ if and only if y_1 is a root of $g(y) = 0$, where $x_1 = 1/y_1$ and

$$y_1 = \sqrt[3]{-g\left(\frac{-c}{3d}\right)} - \frac{c}{3d}$$

By straightforward calculations we find, taking (2) into account,

$$g\left(\frac{-c}{3d}\right) = \frac{27d^2 - c^3}{27d^3}.$$

Hence, the roots of $f(x) = 0$ are given by

$$x = \frac{3d}{(c^3 - 27d^2)^{1/3} - c}. \tag{3}$$

All three roots of $f(x) = 0$ will be obtained by employing, in turn, the three complex cube roots of $c^3 - 27d^2$ in (3).

Case 2. Suppose $c^2 - 3bd$ is not zero. We then let $x = t + a$ and apply to (1) the Taylor expansion,

$$\begin{aligned} f(t + a) &= t^3 + \frac{f''(a)}{2!} t^2 + f'(a)t + f(a) \\ &= t^3 + Bt^2 + Ct + D. \end{aligned}$$

The condition

$$C^2 - 3BD = 0$$

is found to be equivalent to the equation

$$(b^2 - 3c)a^2 + (bc - 9d)a + (c^2 - 3bd) = 0. \tag{4}$$

This equation can be solved for a since the coefficients $(b^2 - 3c)$ and $(bc - 9d)$ cannot vanish simultaneously unless $c^2 - 3bd = 0$. We may therefore employ a root of (4) to reduce the problem to case 1.

A MODERN TRICK

CLAIRE ADLER, New York University

There are old methods for showing that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1,$$

but this one seems extremely simple:

Let

$$\sqrt[3]{2+\sqrt{5}}=a$$

$$\sqrt[3]{2-\sqrt{5}}=b$$

and

$$a+b=x.$$

Then

$$x^3=a^3+b^3+3abx=-3x+4 \tag{1}$$

or

$$x^3+3x-4=0 \tag{2}$$

and the only real root of (2) is

$$x=1.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1016. *Proposed by Norman Anning, University of Michigan*

Find the element of likeness in: (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, (e) putting B, C, D in the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ B & a^3 & 1 & a \\ C & D & a^3 & 1 \end{vmatrix}.$$

E 1017. *Proposed by Harry Furstenberg, New York City*

On the common secant AB of two intersecting circles, O and O', are chosen any two points, C and D, outside of either circle. The tangents CQ and CS are drawn to O' and O, respectively, on one side of AB and the tangents DR and DT are drawn to O' and O on the other side of AB. Prove that QR and ST intersect on AB.

E 1018. *Proposed by B. B. Misra, Ravenshaw College, Cuttack, India*

AB is a line segment, which is divided by the point O in the ratio $m:n$. P is a variable point (r, θ) referred to O as pole and OB as initial line. Subject to the conditions

$$0 \leqq r \leqq a < \min(AO, OB), \qquad 0 \leqq \theta \leqq \pi,$$

find the minimum value of $m^2/AP+n^2/BP$, and also find the position of P for which this minimum value is attained.

E 1019. *Proposed by A. E. Livingston and L. H. Wegner, University of Oregon*

Let $\{f_n(x)\}$ be a sequence of functions continuous on the closed interval $[a, b]$, and let $\lim_{n\to\infty} f_n(x)=f(x)$ uniformly on the open interval (a, b). Then there exists a function $F(x)$ satisfying: (1) $F(x)$ is continuous on $[a, b]$, (2) $F(x)=f(x)$ for $x\epsilon(a, b)$, (3) $\lim_{n\to\infty} f_n(a)=F(a)$ and $\lim_{n\to\infty} f_n(b)=F(b)$.

E 1020. *Proposed by John Disch, Cleveland, Ohio*

A perfectly flexible inextensible chain of uniform density is hanging from two supports at equal heights above the surface of a liquid. If the chain sags into the liquid find the dip in the chain in terms of those factors that affect the problem.

SOLUTIONS

E 967 [1952, 43]. *Correction by J. H. Braun, Washington, D. C.*

In the solution of part (c) replace $[1-5\pi^2/3n^2]$ by $[1-8\pi^2/5n^2]$. For, from (2), it follows that

$$S_n < \pi[1 - (\pi^2/15n^2)(25 - 2\pi^2/n^2)].$$

But, for n an integer greater than 4, we have $25-2\pi^2/n^2>24$, whence

$$S_n < \pi[1 - 8\pi^2/5n^2].$$

Pythagorean Triangles

E 986 [1951, 631]. *Proposed by W. R. Ransom, Tufts College*

Show that there are fifty right triangles whose sides are integers less than 100.

Solution by C. W. Trigg, Los Angeles City College. The hypotenuses of all primitive right triangles are given by $c=m^2+n^2$, where m and n are relatively

prime positive integers of different parity, and $m>n$. We determine every $c<100$ and compute the number (f) of multiples of c less than 100. Thus

n	1	1	1	1	2	2	2	2	3	3	4	4	4	5	5	6
m	2	4	6	8	3	5	7	9	4	8	5	7	9	6	8	7
c	5	17	37	65	13	29	53	85	25	73	41	65	97	61	89	85
f	19	5	2	1	7	3	1	1	3	1	2	1	1	1	1	1

Since $c>a=m^2-n^2$, $c>b=2mn$, and $\Sigma f=50$, there are fifty right triangles with integer sides less than 100.

Also solved by H. T. R. Aude, Leon Bankoff, H. H. Berry, E. R. Bowersox, J. H. Braun, W. E. Briggs, W. E. Buker, P. L. Chessin, D. R. Clutterham, R. J. Cormier, J. E. Darraugh, F. F. Dorsey, William Douglas, J. C. Eaves, Herbert Emich, C. V. Fronabarger, R. A. Gambill and J. D. Neff jointly, Vern Hoggatt, H. B. Hoyle, Jr., Sam Kravitz, Sidney Kravitz, R. Lariviere, Myles McConnon, Jack Minker, Prasert Na Nagara, Elizabeth Napp, Leola Odland, F. D. Parker, C. F. Pinzka, A. P. Rhodes, Azriel Rosenfeld, Charles Salkind, P. T. Schaefer, E. M. Scheuer, R. Z. Vause, Jr., G. W. Walker, and the proposer. Late solutions by W. O. Carter, Barbara Clark, Roxie Eldridge, Dick Hartung, N. C. C. Hu, P. K. Kloeppel, Jane Lassiter, Imogena Lowden, Donald Morrison, F. W. Stallard, Peter Terwey, Jr., W. R. Van Voorhis, W. T. Wolf.

Berry showed that there are 652 Pythagorean triangles with sides less than 1000.

Clutterham solved the problem on the *ORDVAC*, an electronic digital computer that the University of Illinois is building for the Army and which is now in its final testing stages. The vanishing of the function $a^2+b^2-c^2$ was tested for all integral values of a, b, c: $1\leqq a\leqq b\leqq c<100$. The actual computing time was about 8 minutes with another 7 minutes required for the printing of the results.

Douglas called attention to William Sell, *Integral Solutions of* $A^2+B^2=C^2$, this MONTHLY, Oct. 1936, p. 481.

Grid Points in a Strip

E 987 [1951, 631]. *Proposed by W. L. Stamey and J. L. Zemmer, Jr., University of Missouri*

Prove that the interior of the strip, on a cartesian grid, bounded by two arbitrarily close parallel straight lines, contains either no points with integer coordinates or infinitely many such points.

Solution by F. Bagemihl and W. Seidel, University of Rochester. Assume that the strip is not parallel to the y-axis (otherwise the result to be proved is trivial). Then $\epsilon>0$, β, and μ exist, such that the strip is the set of points (x, y) for which $|(y-\mu x)-\beta|<\epsilon$. If μ is irrational, then, according to Kronecker's theorem (see,

e.g., Hardy and Wright, *An Introduction to the Theory of Numbers*, p. 363), this inequality has infinitely many pairs of integral solutions, x, y. If μ is rational, say a/b, and the lattice point (x_0, y_0) satisfies this inequality, then so do the lattice points (x_0+bn, y_0+an), $n=0, \pm1, \pm2, \cdots$.

Also solved by T. M. Apostol, I. A. Dodes, S. H. Gould, Vern Hoggatt, David Mandelbaum, C. S. Ogilvy, Gustave Rabson, and Azriel Rosenfeld.

A Set of Numbers Not All Prime

E 988 [1951, 632]. *Proposed by Leo Moser, Texas Technological College*

Let a, b, x_0 be positive integers and let $x_1=ax_0+b, \cdots, x_n=ax_{n-1}+b$. Prove that x_n cannot be prime for all n.

Solution by N. G. Gunderson, University of Rochester. It is easily proved by induction that $x_n=x_1+(x_1-x_0)(a^n-a)/(a-1)$. Let p be any prime dividing x_1. By Fermat's Theorem p divides $(a^p-a)/(a-1)$, and therefore p divides x_p. But $x_p>x_1\geqq p$, and so x_p is composite.

Also solved by P. M. Anselone, John Christopher, Herbert Emich, C. V. Fronabarger, Emil Grosswald, Prasert Na Nagara, L. W. Neustadt, Joseph Oppenheim, Azriel Rosenfeld, W. Seidel, Peter Terwey, Jr., J. A. Tierney, R. Z. Vause, Jr., and the proposer.

Scissors with a Constant Cutting Angle

E 989 [1951, 632]. *Proposed by C. W. Bruce, Tennessee Polytechnic Institute*

If the angle between the blades of scissors is less than the angle of friction of the metal on the cloth being cut, there will be a little tendency for the cloth to move forward as the scissors are being closed. How can a pair of scissors be constructed so that one blade is straight and there is a constant angle between the cutting edges?

Solution by G. W. Walker, Buffalo, N. Y. For simplicity, place the pivot so that the line of the straight cutting edge will, if continued, pass through it. Then, in any functioning position, the angle between the blades will be the angle between the tangent to the curved cutting edge at the point of action and the radius vector from the pivot to that point. This angle is constant if the curved cutting edge is an equiangular spiral having the pivot as its pole.

Also solved by I. A. Dodes, F. F. Dorsey, Sidney Kravitz, H. L. Lee, David Mandelbaum, J. D. Neff, C. O. Oakley, C. S. Ogilvy, Azriel Rosenfeld, and the proposer. Late solution by Leon Bankoff (who designed a scissors in which both blades are straight).

Spheres of Droz-Farny

E 990 [1951, 632]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere with center at the orthocenter of an orthocentric tetrahedron intersects the lines joining the midpoints of the edges of the tetrahedron, then the 24 distances of the vertices from the intersections on those lines joining the

midpoints of the edges issuing from these vertices are all equal. (The analogous property for the triangle was announced by Steiner and proved by Droz-Farny. See Johnson, *Modern Geometry*, p. 256.)

Editorial Note. The proof of the analogous property for the triangle as given in Johnson's book carries over almost verbatim for the tetrahedron.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4488. *Proposed by George Grossman, DeWitt Clinton High School, New York City*

Using the symbolic representation

$$c^{(3)} \equiv 8m_0 + 12m_1 + 6m_2 + 1m_3$$

to indicate that a cube of dimensionality 3 possesses 8 vertices (dimensionality 0), 12 manifolds of dimensionality one, etc., then a tesseract (hypercube of dimensionality four) can be represented symbolically by

$$c^{(4)} \equiv 16m_0 + 32m_1 + 24m_2 + 8m_3 + 1m_4.$$

If a hypercube of dimensionality n is represented by

$$c^{(n)} \equiv a_{n0}m_0 + a_{n1}m_1 + \cdots + a_{nn}m_n,$$

determine the coefficients a_{ij}.

4489. *Proposed by D. J. Newman, Harvard University*

Sum the series

$$\sum_{n=1}^{\infty} \frac{(n/e)^{n-1}}{n!}.$$

4490. *Proposed by L. M. Blumenthal, University of Missouri*

Let S be a set of more than $2n$ pairwise distinct (non-null) vectors of Euclidean n-space E_n. If each vector of E_n is expressible as a linear combination of vectors of S with non-negative coefficients, then S contains a subset R of at most

$2n-1$ vectors such that R has this same property. (This is an extension of no. 4395 [1952, 46]).

4491. *Proposed by C. S. Venkataraman, Trichur, South India*

If ω denotes the Brocard angle of a triangle ABC, prove that

(i) the sides are equal when $\cot \omega = \sqrt{3}$.

(ii) the square of the lengths of the sides, a^2, b^2, c^2, are in arithmetic progression when $\cot \omega = 3 \cot B$.

4492. *Proposed by G. Lumer, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

A real function $f(x)$ is said to be locally increasing at the point x_0 if there exists a neighborhood U of x_0 such that for $x \in U$,

$$\frac{f(x) - f(x_0)}{x - x_0} \geqq 0.$$

Show that there exist functions $f(x)$, defined and continuous on a closed interval $[a, b]$, locally increasing on a set D, everywhere dense on $[a, b]$, but such that $f(b) < f(a)$.

SOLUTIONS

Orthocentric Tetrahedron Bordered by Isosceles Tetrahedrons

4417 [1950, 691]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) The necessary and sufficient condition that an orthocentric tetrahedron $T \equiv ABCD$ can be bordered by isosceles tetrahedrons $T_a \equiv A_1BCD$, $T_b \equiv B_1CDA$, $T_c \equiv C_1DAB$, $T_d \equiv D_1ABC$ is that all the face angles of T be acute. (2) The tetrahedrons T and $T_1 \equiv A_1B_1C_1D_1$ are orthologic. (3) The lines AA_1, BB_1, CC_1, DD_1 are four hyperbolic lines. (4) The sum of the squares of the areas of the faces of T_a, T_b, T_c, T_d is equal to the sum of the squares of the products of the opposite edges of T.

Solution by the Proposer. The following hints contain all the essentials of the argument.

(1) The face angles of any isosceles tetrahedron are all acute.

(2) The perpendiculars from A_1, B_1, C_1, D_1 on the opposite faces of T go through the points symmetrical to the orthocenters of these faces with respect to their circumcenters. Hence they are concurrent at a point symmetrical to the orthocenter H of T with respect to the circumcenter O of T. It follows that the perpendiculars from A, B, C, D on the faces $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ of T_1 are also concurrent.

(3) The lines AA_1, BB_1, CC_1, DD_1 intersect OH.

(4) In an orthocentric tetrahedron the sum of the squares of the areas of the faces is equal to

$$\tfrac{1}{4}[(aa')^2 + (bb')^2 + (cc')^2],$$

where $a = BC$, $a' = AD$, *etc.*

Editorial Note. The translator, W. E. Byrne, supplies the following references from N. A. Court, *Modern Pure Solid Geometry.*

Art. 212. Theorem. The four altitudes of an orthocentric tetrahedron are concurrent.

Art. 213. Definition. The point common to the four altitudes of an orthocentric tetrahedron is called the orthocenter of the tetrahedron.

Art. 216. Theorem. The altitudes of an orthocentric tetrahedron pass through the orthocenters of the respective faces of the tetrahedron.

Art. 290. Definition. A tetrahedron in which each edge is equal to its opposite will be referrred to as an isosceles tetrahedron.

Art. 309. Theorem. The foot of the altitude dropped from a vertex of an isosceles tetrahedron upon the corresponding face is the symmetric of the orthocenter of this face with respect to its circumcenter.

Determinant Evaluation

4418 [1950, 692]. *Proposed by A. C. Aitken, Edinburgh, Scotland*

Evaluate the determinant

$$(1) \qquad a_n = \begin{vmatrix} b_1 & -1 & 0 & \cdots & 0 \\ b_2 & b_1 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & 1-n \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \end{vmatrix} \quad \text{where} \quad (2) \quad b_n = \frac{n^n}{n!}.$$

Solution by J. S. Frame, Michigan State College. In general, to obtain a relation between the coefficients a_n and b_n in (1), without taking account of (2), we may set $a_0 = 1$, $a_1 = b_1$, and define the two functions y and z by the power series

$$(3) \qquad y = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad z = \sum_{n=1}^{\infty} b_n \frac{x^n}{n}.$$

Upon expanding the determinant (1) by cofactors of its last row we obtain the recurrence relation

$$(4) \qquad n\frac{a_n}{n!} = a_0 b_n + \frac{a_1}{1!} b_{n-1} + \frac{a_2}{2!} b_{n-2} + \cdots + \frac{a_{n-1} b_1}{(n-1)!},$$

which is equivalent to the differential equation $y'=yz'$, whose solution, satisfying $y=1$ when $x=z=0$, is

$$y = e^z. \tag{5}$$

Hence, using the coefficients b_n defined in (2), our problem is to find the coefficient a_n of $x^n/n!$ in the expansion of e^z when

$$z = \sum_{k=1}^{\infty} \frac{(kx)^k}{k \cdot k!}. \tag{6}$$

If in (6) we make the substitution

$$x = te^{-t}, \tag{7}$$

we find

$$\begin{aligned} \frac{z}{t} &= \sum_{k=1}^{\infty} \frac{(kt)^{k-1}e^{-kt}}{k!} = \sum_{m,k=1}^{\infty} (-1)^{m-k}\binom{m}{k}\frac{(kt)^{m-1}}{m!} \\ &= 1 + \sum_{m=2}^{\infty} \frac{t^{m-1}}{m!}\left[\sum_{k=0}^{m} (-1)^{m-k}\binom{m}{k}k^{m-1}\right]. \end{aligned} \tag{8}$$

The expression in brackets vanishes for $m>1$, since it is the coefficient of $t^{m-1}/(m-1)!$ in the expansion of $(e^t-1)^m$. Hence the right member of (8) reduces to 1, and we have $z=t$ and from (7) $x=z\,e^{-z}$. It follows from (5), (3), and (2) that

$$y = e^z = z/x = \sum_{n=0}^{\infty} b_{n+1}x^n/(n+1) = \sum_{n=0}^{\infty} (n+1)^{n-1}x^n/n! \tag{9}$$

By comparing coefficients of $x_n/n!$ in (3) and (9) we obtain the required formula for the determinant a_n in (1), namely,

$$a_n = (n+1)^{n-1}. \tag{10}$$

Also solved by N. J. Fine, G. P. Henderson, O. E. Stanaitis, G. Szegö, and the Proposer.

Editorial Note. Fine obtains recurrence formula (4) above (with the values of b_k introduced) and recognizes it as a familiar formula for $u_n \equiv (n+1)^{n-1}$. In fact, recalling a result due to Abel,

$$(x+y+nz)^n = y\sum_{j=0}^{n}\binom{n}{j}[n+(n-j)z]^{n-j}(y+jz)^{j-1},$$

and placing $x=y=z=1$, he has

$$u_{n+1} = \sum_{j=0}^{n}\binom{n}{j}(n-j+1)^{n-j}u_j.$$

If j is replaced by $n-k$, it is seen that u_n satisfies the same recurrence formula that a_n does. Since the initial conditions are also the same, $a_n = u_n$ for all n.

The Proposer obtains the evaluation by induction, using the determinant $\Delta(t)$ formed from the given determinant by replacing each $k^k/k!$, in the last row only, by $t^{k-1}/(k-1)!$. Then $\Delta(n)$ is the given determinant. It is shown that $\Delta(t)$ equals $(t+1)^{n-1}$ by obtaining

$$\Delta(n-1) = n^{n-1},\ \Delta'(n-2) = (n-1)^{n-1},$$
$$\Delta''(n-3) = (n-1)(n-2)^{n-2}, \cdots, \Delta^{(n-1)}(0) = (n-1)!$$

Stanaitis finds the proposed determinant as a special case of

$$\begin{vmatrix} c_1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2c_2 & c_1 & -2 & \cdot & \cdot & \cdot & \cdot \\ 3c_3 & 2c_2 & c_1 & -3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = n! \sum \frac{c_1^{e_1} c_2^{e_2} \cdots c_n^{e_n}}{e_1! e_2! \cdots e_n!},$$

where $$e_1 + 2e_2 + \cdots + ne_n = n.$$

See Muir-Metzler, *A Treatise on the Theory of Determinants*, 1930, p. 719.

Concerning a Theorem of McCoy

4419 [1950, 692]. *Proposed by L. G. Jones, University of Oregon and Alex Rosenberg, University of Toronto*

In a paper, *Remarks on Divisors of Zero*, this MONTHLY [1942, 286–295], N. H. McCoy stated and proved the following theorem: Let $f(x)$ be an element of $R[x]$, where R is a commutative ring. Then $f(x)$ is a divisor of zero in $R[x]$ if and only if there exists a nonzero element c of R such that $cf(x) = 0$.

Show that, if R is not assumed to be commutative, the theorem no longer holds.

Solution by Louis Weiner, University of Chicago. A counter example is provided if for R we choose the set of all 2 by 2 matrices. We set

$$f(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then if $\mathbf{c}f(x) = 0$, where $\mathbf{c} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ c & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b & b \\ d & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

whence $a = c = 0$, $b = d = 0$. However if

$$g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have $g(x)f(x)=0$.

Note. The theorem also fails if R is not assumed to be associative; for let R be the Jordan algebra with basis $1, u_1, u_2$ where $u_1^2=u_2^2=1$, $u_1u_2=u_2u_1=0$, and let $f(x)=u_1x+u_2$. Then if $\mathbf{c}f(x)=0$ where $\mathbf{c}=a+bu_1+cu_2$, we have $\mathbf{c}u_1=au_1+b=0$, $a=b=0$; $\mathbf{c}u_2=au_2+c=0$, $c=0$; $\mathbf{c}=0$. However, if $g(x)=u_2x-u_1$, then $f(x)g(x)=-u_1^2+u_2^2=-1+1=0$.

Also solved by I. H. Rose and by each of the Proposers.

Summation of Tangents

4420 [1951, 42]. *Proposed by F. S. Acton, National Bureau of Standards, Institute for Numerical Analysis*

Evaluate

$$g(m)=\sum_{j=0}^{m-1}(-1)^j\tan\left[\frac{\pi}{2}\left(\frac{2j+1}{2m}\right)\right].$$

I. *Solution by J. B. Rosser, Cornell University, and the Proposer.* Consider the function $f(z)=-2m\tan z/\cos 2mz$. This has poles along the real axis,

$$\begin{cases}\text{from the cosine, at } 2mz=(j+\frac{1}{2})\pi,\ j=0,1,\cdots.\\ \text{from the tangent, at } k\pi/2,\ k=1,3,\cdots.\end{cases}$$

If we evaluate $[\int_C f(z)\,dz]/2\pi i$, where C is the rectangle in the z-plane with vertices $\pm iy_0$, $\pi\pm iy_0$, the cosine residues give rise to twice $g(m)$ because of the antisymmetry of both tangent and sine with respect to $\frac{1}{2}\pi$. The residue at $\frac{1}{2}\pi$ is $-2m/-\cos 2m(\frac{1}{2}\pi)$ or $2m(-1)^m$.

Because of the periodicity of both tangent and cosine, $f(z)$ is identical for $x=0$ and $x=\pi$, and the contributions along the horizontal parts of C may be made to vanish by removing these sides of the rectangle to $\pm\infty$, causing the whole integral to become zero.

Thus $2g(m)+(-1)^m2m=0$ or $g(m)=(-1)^{m+1}m$.

II. *Solution by F. Underwood, University of Nottingham, England.* Since $\tan\theta=-\tan(\pi-\theta)$, $g(m)$ may be written

$$g(m)=\sum_{k=0}^{m-1}\tan\frac{(4k+1)\pi}{4m}.$$

The equation

$$\tan\alpha=\tan m\theta=\frac{{}_mC_1t-{}_mC_3t^3+\cdots}{1-{}_mC_2t^2+\cdots},\qquad t=\tan\theta, \tag{1}$$

is satisfied if $\tan\theta=\tan[(k\pi+\alpha)/m]$. Hence (1), with $\alpha=\pi/4$, is an equation whose roots are given by the m terms whose sum is $g(m)$. Now equation (1) reduces to

$$t^m+mt^{m-1}+\cdots+(-1)^{m/2}=0,$$
$$t^m-mt^{m-1}+\cdots+(-1)^{(m+1)/2}=0,$$

according as m is even or odd. Accordingly, the sum of the m roots, namely $g(m)$, equals $-m$ or $+m$, respectively. Both cases are included in the result: $g(m)=(-1)^{m+1}m$.

Also solved by Jewell H. Bushey, C. E. Lemke, Max LeLeiko, and O. E. Stanaitis.

Irreducible Polynomials

4421 [1951, 42]. *Proposed by N. C. Arkeny, Princeton University*

Prove that there are infinitely many polynomials of degree n with integer coefficients, the first being unity, which have n real roots and are irreducible in the field of rational numbers.

I. *Solution by William Scott, University of Kansas.* Let p be any prime, and let

$$f(x) = p + \sum_{j=1}^{n} (x - 3jp).$$

Then for $j=0, 1, \cdots, n$,

$$f((3j+1)p)\begin{cases} >0 & \text{if } n-j \text{ is even,} \\ <0 & \text{if } n-j \text{ is odd.} \end{cases}$$

Thus $f(x)$ has n real roots. It is irreducible by Eisenstein's criterion.

II. *Solution by P. T. Bateman and L. Schoenfeld, University of Illinois.* If $a_1, a_2, \cdots, a_n$ are any integers such that $a_k \geqq a_{k+1}+2$ $(k=1, \cdots, n-1)$, then

$$f(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - 1$$

is irreducible and has n roots. The irreducibility is well known and easy to prove (Pólya and Szegö, *Aufgaben and Lehrsätze aus der Analysis*, Part VIII, Problem 121.) To see that $f(x)$ has n real roots suppose $n \geqq 2$ and choose integers $b_0, b_1, \cdots, b_n$ such that

$$b_0 > a_1 > b_1 > a_2 > \cdots > a_{n-1} > b_{n-1} > a_n > b_n.$$

Then if $0 \leqq 2j \leqq n$ the product $(b_{2j}-a_j)(b_{2j}-a_2)\cdots(b_{2j}-a_n)$ is a positive integer greater than unity and thus $f(b_{2j})>0$. Also if $1 \leqq 2j+1 \leqq n$ clearly $f(b_{2j+1})<0$. Hence $f(x)$ has n real roots.

Also solved by Helen C. Arens, Alfred Brauer (two solutions), N. J. Fine, J. B. Kelly, A. H. Kruse, D. R. Morrison, Leo Moser, Morris Newman, P. A. Samet, Olga Taussky-Todd, J. L. Ullman, V. J. Varineau, Morgan Ward, and the Proposer.

The n Kings Problem

4422 [1951, 42]. *Proposed by Jacques Dutka, Rutgers University*

n elements numbered 1, 2, $\cdots$, n, stand in order. In how many ways can

they be rearranged in a line so that no two are together which originally stood together?

Solution by the Proposer. Reference is made to I. Kaplansky, *The Asymptotic Distribution of Runs of Consecutive Elements*, Annals of Math. Statistics, vol. 16 (1945) pp. 200–203. If r denotes the number of instances in which a succession $i, i+1$ or $i+1, i$ occurs in a permutation of $1, 2, \cdots, n$, then the corresponding probability is given by Kaplansky as

$$P(n, r) = \frac{2^r e^{-2}}{r!}\left[1 - \frac{r^2 - 3r}{2n} + \frac{r^4 - 8r^3 + 9r^2 + 22r - 16}{8n(n-1)}\right] + O(n^{-3}).$$

The distribution of r is asymptotically a Poisson distribution, and therefore the answer to the proposed problem may be written as

$$n!P(n, 0) = n!\left\{e^{-2}\left[1 - \frac{16}{8n(n-1)}\right] + O(n^{-3})\right\}.$$

For example, if $n=10$, $n!\,P(n, 0)$ is almost $(10!)(0.132)$.

Note by Leo Moser, University of Alberta. This problem was proposed previously as E 777 [1948, 160]. As noted there, a solution is found by I. Kaplansky in his paper *Symbolic Solution of Certain Problems in Permutations*, Bull. Amer. Math. Soc., vol. 50 (1944), pp. 911–913. This solution, however, is in symbolic form and is not easily expressed by a simple explicit formula. As Kaplansky points out, the problem may be formulated as follows: In how many ways can n kings be placed on an n by n chess board, one on each row and column, so that no two attack each other?

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any other editors or officers of the Association.

Introduction to Number Theory. By Trygve Nagell. New York, John Wiley and Sons Co. 1951. 309 pages. $5.00.

The literature of elementary number theory contains a vast number of topics which the beginning student may study with profit and pleasure. It is therefore not surprising that almost all of the excellent texts which are available on number theory contain important material not to be found in the other texts. The

present book is no exception in this respect. With the characteristic enthusiasm of a devotee of the queen of mathematics the author has provided the reader with a brief introduction to some of the most important achievements of elementary number theory. Among the several results and methods published for the first time in book form is Selberg's elementary proof of the prime number theorem.

An idea of the scope of the book may be obtained from the chapter titles: 1. Divisibility; 2. On the distribution of primes; 3. Theory of congruences; 4. Theory of quadratic residues; 5. Arithmetical properties of the roots of unity; 6. Diophantine equations of the second degree; 7. Diophantine equations of higher degree; 8. The prime number theorem.

The book gives a comprehensive account of classical number theory and includes a number of recent results. Throughout the text historical comments and statements of modern results are liberally interspersed; these are certain to vivify the student's understanding of the subject. There are 180 exercises. Many of these problems are by no means routine or elementary in nature and will challenge the ingenuity of the most ambitious student.

Some of the more notable features of the book are the following. Included in Chapter 1 is Niven's recent (1947) proof of the irrationality of π. In Chapter 3 the important remainder theorem of Aubry-Thue and its generalization by Scholz are proved. The name of Aubry is, however, not mentioned. These theorems are employed in Chapter 6 to derive in an elegant manner a number of theorems about quadratic partitions of primes. In particular the classical result that every prime p of the form $4n+1$ is the sum of two squares is thus established. Chapter 5 contains an interesting introduction to the Gauss sums and cyclotomic polynomials. A theorem of Bauer on the prime divisors of certain polynomials is used to prove that if n is an integer, there are infinitely many primes which are $\equiv -1 \pmod{n}$.

Chapters 6 and 7 on Diophantine analysis constitute an admirable introduction to a difficult subject. After presenting a concise and careful exposition of the classical theory the author, with a master's hand, guides the reader directly to the fringes of unexplored territory. Here theorems associated with the names of Kummer, Maillet, Mordell, Siegel, Skolem, Thue, Weil and the author himself may be found. An outstanding result of Vandiver in connection with Fermat's last theorem is cited. Unfortunately, there is no reference to the name of Vandiver.

On page 52 the statement is made that the Mersenne number $M_{127}=2^{127}-1$ is the largest number known at present to be a prime. This number which enjoyed its position of sovereignty during the entire first half of the present century has this year toppled from its throne. Professor D. H. Lehmer has kindly informed the reviewer that some indefatigable computers have recently unearthed larger primes. In Cambridge, England, the Edsac machine has been used to verify by the converse of Fermat's theorem the primality of the number $180M_{127}^2+1$. In Allier, France, A. Ferrier has used a desk calculator to identify

the number $(2^{148}+1)/17$ as a prime. One recalls with amusement that in 1811 Peter Barlow wrote in his "Theory of Numbers" that " . . . the Mersenne number $M_{31}=2^{31}-1$ is the largest number known at present to be prime, and probably the greatest that ever will be discovered; for, as they are merely curious without being useful, it is not likely that any person will attempt to find one beyond it."

The last chapter is the author's *pièce de résistance.* Here is presented for the first time in any book Selberg's sensational elementary proof of the prime number theorem. The basis of the proof is the following asymptotic formula obtained by Selberg in 1948:

$$\sum_{p\leq x} (\log p)^2 + \sum_{pq\leq x} \log p \log q = 2x \log x + O(x),$$

where p and q run over the primes. This formula is itself a consequence of the prime number theorem; Selberg's ingenious proof is, however, completely elementary. From the formula there are several ways to deduce the prime number theorem. The author follows an exposition given by van der Corput and based on notes of some lectures given by Erdös in Amsterdam in 1948. For the sake of historical accuracy the reviewer regrets that the author does not even allude to the role played by Erdös in Selberg's discovery of his first proof. In his original proof Selberg made use of the following result by Erdös: For an arbitrary, positive fixed number δ, there exists a $K(\delta)>0$ and an $x_0=x_0(\delta)$ such that for $x>x_0$, there are more than $K(\delta)x/\log x$ primes in the interval from x to $x+\delta x$.

The appearance of this book has unquestionably solved for many teachers of number theory the perplexing problem of finding a text suited to the needs of the classroom.

A. L. WHITEMAN
University of Southern California

Les Récréations Mathématiques (Parmi les Nombres Curieux). By Victor Thébault. Gauthier-Villars, Paris, 1952. 6+298 pages. 2500 fr.

The author has provided in this volume a compendium of about a hundred fifty arithmetical problems of types already familiar to readers of *Mathesis* and of the Problem Departments of this MONTHLY. A large number of the problems and much of the discussion have been published previously. In particular pp. 188–195 duplicates two recent articles in this MONTHLY, [1949, 443–448], [1951, 101–103], although the results are considerably extended in the pages which follow.

As the title indicates, the problems are mostly recreational in character and deal with curiosities among the integers. Specially featured are: squares, cubes, products, *etc.*, which have unusual digit patterns, or can be written with the digits 0, 1, · · · , 9 used just once each; numbers all of whose powers have the same set of terminal digits; sums of selected elements in addition and multiplication tables; numerical triangles satisfying various requirements; and different

systems of notation in which such properties may be exhibited. For some of the problems fairly complete solutions are provided, for others, briefer hints, and for many more, mere indications of the results. The methods employed are entirely elementary, mainly arithmetic and elementary algebra (and trigonometry where applicable), together with a little number theory (through Fermat's theorem and the solution of the Pell-Fermat equation). The solutions are frequently very ingenious; ingenious, too, is the discovery of so many unusual problems which will yield to the available methods of treatment.

The author makes an eloquent appeal for the significance and use of mathematical recreations both because of the high place accorded them by many of the leading mathematicians and also because of their undoubted value as an instrument for fostering a love and appreciation for mathematics in the developing student mind.

The value of the book would have been enhanced by better organization of material. While it is true that assorted problems are difficult to index and provide with cross-references, yet more could have been done to help the reader locate problems of any desired type. A large number of misprints appear, although they seldom obscure the argument and will confuse the reader for only a moment or two.

Altogether this is a book well worth reading, and it will be an unusual reader who does not return again and again to renew his acquaintance with some problem that holds out a special appeal to him.

Several tables are provided, such as the squares, in the decimal system, having all distinct digits and the squares of six or less digits in all systems of notation from 2 to 12 inclusive. A short presentation of the theory of the Pell-Fermat equation and a note on numerical triangulation by M. A. Buquet are included as an appendix.

E. P. STARKE
Rutgers University

NEW BOOKS RECEIVED

Conformal Mapping. By Zeev Nehari. New York, McGraw-Hill Book Company, 1952. viii+396 pp. $7.50.

Business Mathematics. By W. F. Cassidy and C. C. Robusto. New York, Prentice-Hall, 1952. vii+109 pp. $4.75.

The Design and Analysis of Experiments. By Oscar Kempthorne. New York, John Wiley and Sons, 1952. xix+631 pp. $8.50.

Statistical Calculation for Beginners. By E. G. Chambers. New York, Cambridge University Press, 1952. 168 pp. $2.50.

Mathematical Tables, vol. 1, 3rd ed., Circular and Hyperbolic Functions. By the British Association for the Advancement of Science, issued by the Royal Society. England, Cambridge University Press, 1952. 40+72 pp. $3.50. (Available at American branch of Cambridge University Press, New York.)

Classical Mechanics. By D. E. Rutherford. Edinburgh and London, Oliver and Boyd, 1951. (Available at Interscience Press, New York.) viii+200 pp. 10/6 net.

Partial Differentiation. By R. P. Gillespie. Edinburgh and London, Oliver and Boyd, 1951. (Available at Interscience Press, New York.) viii+105 pp. 6/—net.

Nomography and Empirical Equations. By L. H. Johnson. New York, John Wiley and Sons, 1952. ix+150 pp. $3.75.

A University Algebra. By D. E. Littlewood. London, England, William Heinemann, Ltd., 1950. 8+292 pp. 25s net.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

Mathematics Club, Columbia College

The *Mathematics Club of Columbia College* presented the following programs during 1950–51:

A theorem of Kronecker, by Mr. Cornelius W. Langley

The gamma function, a series of two lectures by Professor Claude Chevalley

Topology, a series of three lectures by Mr. Marvin P. Epstein

The Laplace transform, by Professor Walter Strodt

Euclidean geometry, by Professor E. R. Lorch

Mathematical models in statistics, a series of two lectures by Mr. Daniel Slotnick.

Officers elected for 1951–52 are: President, Elliot Auerbach; Vice-President and Secretary, David Berkowitz; Vice-President and Treasurer, Charles Kaufman.

Mathematics Club, University of Buffalo

The *Mathematics Club of the University of Buffalo* held six regular meetings during 1950–51. The following papers were presented:

An approximation method for determining dielectric constants, by Roy Hendricks, Cornell Aeronautical Laboratories

A discussion of problem 4401, the American Mathematical Monthly, and a generalization, by Donald O. McKay

A construction problem, by Eleanor S. Krivian

Paradoxes of the infinite as encountered in conditionally convergent series, by Joseph E. Kist

Numbers on a circle, by Michael J. Peinkofer

Mind-reading tricks with numbers, by Donald O. McKay.

An award from the American Chemical Rubber Co. was given to Pearl M. Snitker for outstanding work in freshman mathematics.

Other events included a special program for high school students, a Christmas dinner, and a picnic in June.

Officers elected for 1951–52 are: President, Leonard Schaer; Vice-President, David Kline; Secretary, Irma Rosen; Treasurer, Beverly Dale; Refreshments Committee, Rita Mochan and Jane Riley.

Kappa Mu Epsilon, Central Michigan College

The *Michigan Beta* chapter of *Kappa Mu Epsilon* at *Central Michigan College* held six meetings during 1950–51. The following papers were presented:

Mathematics and games of chance, by Lloyd Trinklein

Mathematics and religion, by Wayne Atkins.

Officers elected for 1951–52 are: President, Richard Little; Vice-President, Paul Buchholz; Treasurer, Donald McPhee; Secretary, Joseph Pipoly.

Pi Mu Epsilon, University of Alabama

The *Alabama Alpha* chapter of *Pi Mu Epsilon* held four program meetings during 1950–51, in addition to a Christmas party, a spring picnic, and two combined business meetings and initiation ceremonies. The following papers were presented:

Methods of reducing curves to unit curves in integration, by Mr. Herbert Diekhans

Statistics, common sense and epistemology, by Professor O. L. Lacey

The origin of mathematics, by Mr. Hassel Palmer

Theorem of Helly, by Mr. B. Vanderburg.

For the third consecutive year, the chapter sponsored a campus-wide competitive examination in mathematics. Prize winners were Donald A. Swenson, Walter B. Mitchell, Billy Letson, J. J. Samuels, and Robert Woltman.

The officers elected for 1951–52 are: Director, Donald A. Swenson; Vice-Director, Ying Victor Wu; Secretary, Betty Campbell; Treasurer, Louis Jaffe; Librarian, Dr. J. D. Mancill; Social Chairman, Elizabeth Cathey.

Pi Mu Epsilon, Michigan State College

The *Michigan Alpha* chapter of *Pi Mu Epsilon* held thirteen meetings during 1950–51, including business meetings, program meetings, initiations, the annual winter banquet, and the spring picnic. Topics presented were as follows:

Dissection problems on squares, by Dr. Stewart

Applications of mechanics to geometry, by Dr. L. M. Kelly

Soap film problem, by James Powell

Mathematics as a career, panel discussion by Dr. Stewart, Dr. Powell, Dr. Wells, Dr. Katz, and Dr. Herzog

The problem of the queens, by Dr. Herzog

Education in Puerto Rico, by Dr. Grove

The modern theory of dimension, by Dr. Nordhaus

Series and sequences, by Professor George Piranian, University of Michigan.

The L. C. Plant Awards, presented annually to the students who have in the past year contributed the most to mathematics through scholarship, interest in mathematics, and help to the mathematics department, were awarded to Charles Parker and R. Douglas Behr.

Officers for 1951–52 are: President, Paul Buben; Vice-President, Douglas Behr; Secretary, Henry Leonard; Treasurer, Charles Parker; Permanent Secretary, James Powell; Faculty Advisors, Dr. L. M. Kelly and Dr. J. E. Powell.

Mathematics Club, Stanford University

The following papers were presented to the *Stanford Mathematics Club* in 1950–51:

Report on the International Congress, by Professor G. Szegö

Some elementary topics in number theory, by Professor C. C. Olds

Some elementary games, by Professor D. Blackwell

Applications of calculus of variations to the vibrating membrane, by Mr. Joseph Carter

Some topics on the theory of large numbers, by Professor J. G. Van der Corput

Some theorems in Boolean algebra, by Mr. Gordon Hedrick

Mechanical linkages, by Mr. William Firey

The Bush differential analyzer, by Mr. Lloyd Bell.

Kappa Mu Epsilon, Southwestern Louisiana Institute

The *Louisiana Beta* chapter of *Kappa Mu Epsilon* was installed at *Southwestern Louisiana Institute*, Lafayette, on May 22, 1951, with Professor Z. L. Loflin acting as installing officer.

The officers are: President, Tuney Arceneaux; Vice-President, Luiz Figueiredo; Secretary, Elizabeth Ann Roberts; Treasurer, Charles Breaux; Corresponding Secretary and Faculty Sponsor, Merlin M. Ohmer.

Undergraduate Mathematics Club, University of Rochester

The following talks were given at the meetings of the *Undergraduate Mathematics Club of the University of Rochester* during 1950–51:

Mathematics and card tricks, by Robert Wadsworth

Topology, by Manuel Paxia

Mathematics in music, by Paul Schaefer

Mathematical education in Germany, by Dietic Gaier, Göppingen, Germany

I. B. M. machines, by Kathleen Berger

Careers in mathematics, by Charles Meadow.

The award for the best talk was given to Paul Schaefer.

Mathematics Club, Oberlin College

The following papers were presented during 1950–51 at meetings of the *Mathematics Club of Oberlin College:*

Problemes de Rencontre, by Benson Scheff

Cryptology, by Marna Kunstman

Something new under the sun, by James Moore

The meaning of curves, by Myra Parker

Infinity and trans-finite numbers, by Jeanne Taylor

Mathematical methods in economics, by Professor Kenneth Roose

Bridge, by Robert Whitney

Communication theory, by Richard Hartong

Perspective in art, by Professor Seymour Slive.

Professor Frank Ebersole spoke at the annual spring banquet on *Symbolic Logic.*

Officers elected for 1951–52 are: President, Robert Whitney; Vice-President, Edward Assmus; Secretary-Treasurer, Marion Goss; Publicity Chairman, John White; Social Chairmen, Elizabeth Doan and Louise Mayne; Faculty Advisor, Professor Alan Anderson.

Kappa Mu Epsilon, Central Missouri State College

The *Missouri Beta* chapter of *Kappa Mu Epsilon* held regular monthly meetings during 1950–51. The following talks were presented by members of the chapter:

Proofs for $\lim_{x \to 0} (\sin x)/x = 1$, by Donald Jones

Congruences, by Leonard Molotsky

Trigonometric functions expressed in terms of exponential functions, by Wendell McGuire

Numbers that mean too much, by Margaret Honn

Pythagorean number triples, by Kyriakos Lypirides

History and methods of computing pi, by Martin Tempel

The gamma function, by Keith Stumpff

Non-uniform motion of a piston, by Charles Sigrist

Report on the eighth biennial convention of Kappa Mu Epsilon, by Keith Stumpff.

Mr. Robert Jones, a student guest speaker, gave an illustrated talk on *Geometry in architecture.* Mr. F. W. Straulman of the Missouri Inspection Bureau was the speaker at the annual banquet; his talk was entitled *Fire insurance rating.*

Officers elected for 1951–52 are: President, Charles Sigrist; Vice-President, Donald Jones; Treasurer, Wendell McGuire; Faculty Sponsor and Corresponding Secretary, Professor Loren W. Akers.

Pi Mu Epsilon, University of California

The following talks were given in 1950–51 at the six program meetings of the *California Beta* chapter of *Pi Mu Epsilon:*

Large scale high-speed computers, by Professor D. H. Lehmer

Inner and outer development of mathematics, by Professor Abraham Seidenberg

Algebraic identities involving sums of squares, by Professor Raphael M. Robinson

Codification of geometry, by Dr. Ting K. Pan

Post algebra, by Mr. Ralph Willoughby

Gambling and mathematics, by Professor Michel Loève.

The officers elected for 1951–52 are: Director, William Noh; Vice-Director, Vivian Koblick; Secretary, Susan Chakmakjian; Treasurer, Newman H. Fischer Jr.; Librarian, Sarah Hallum.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Annual Meeting of the American Society for Engineering Education will be held at Dartmouth College, Hanover, New Hampshire, on June 23–27, 1952. The program of the Mathematics Division is as follows:

First Session, June 23, 2:00 P.M., "Operational Calculus for Engineering Students," by I. F. Ritter; "The Straight-Ahead Transport Equation," by J. E. Wilkins, Jr.; "What is Knowledge?", by G. C. Vedova; "The Study of a Cantilever Beam Under an Air Blast," by R. R. Stoll;

Second Session, June 24, 2:00 P.M., "Dual Numbers and Vectors," by W. E. Restemeyer; "Teaching With Color," by W. C. Krathwohl; "Mathematics, Machines and Error Analysis," by K. S. Miller;

Third Session, June 25, 2:00 P.M., Business Meeting of the Division; "The Teaching Aids Reviewing Committee Reports on Mathematics," by E. A. Whitman; "The General Electric Mathematics Fellowship Program for Secondary School Teachers," by E. B. Allen; "Engineering Applications of the Mathematical Theory of Probability," by Samuel Goldberg; "An Integrated Approach to Basic Mathematics," by W. R. Van Voorhis.

Dr. R. S. Burington will speak at the dinner on June 25 on the subject:

"What Approaches Would I Use, Were I to Teach Mathematics Again?"

For further information write to: Professor H. M. Gehman, chairman of the Mathematics Division, University of Buffalo, Buffalo 14, New York; or, to Professor J. H. Zant, secretary of the Division, Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma.

STATISTICAL SUMMER SESSION

The Department of Statistics and the Statistical Laboratory in cooperation with the Department of Mathematics and the Department of Industrial Engineering of the Virginia Polytechnic Institute will conduct a special statistical summer session, July 29 to August 15, 1952. The program will be for graduate students, research workers, and technicians in government and industry. Special offerings will be given in the statistics of taste testing, bio-assay, sampling and in engineering research and production. For further details write the Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia.

MATHEMATICS INSTITUTE AT THE UNIVERSITY OF HOUSTON

The University of Houston, Texas, will hold its Second Mathematics Institute on July 28–31, 1952. This Institute is being sponsored by the College of Education, the Mathematics Department of the College of Arts and Sciences, and the College of Business Administration.

Specialists in the field of mathematics teaching who will participate in the Institute program are: Dr. Edwina Deans, elementary supervisor, Arlington, Virginia; Dr. H. F. Spitzer, principal of the University Elementary School and associate professor of education, University of Iowa; and Dr. J. R. Mayor, chairman of the Department of Education and associate professor of mathematics, University of Wisconsin. Mr. Martin Wright, associate professor of mathematics, University of Houston, will address one of the general sessions. A mathematics laboratory for teachers of junior and senior high school mathematics will be directed by Miss Ida May Bernhard, supervisor, College Laboratory School, Southwest Texas State Teachers College, San Marcos, Texas.

All meetings will be held in air-conditioned rooms; residence can be secured in air-conditioned dormitories.

For further information, write Joyce Benbrook, Box 554, University of Houston, Houston 4, Texas.

MATHEMATICS INSTITUTE AT LOUISIANA STATE UNIVERSITY

Louisiana State University will hold its third annual session of the Mathematics Institute on June 15–21, 1952.

There will be discussion groups in algebra, geometry, arithmetic, junior high school mathematics and enrichment materials. Also, there will be a geometry laboratory. The discussions will be led by experts in these fields. In addition there will be lectures given by outstanding people in mathematics and related fields.

Excellent rooms and meals will be provided on the campus at reasonable rates. Plan a June trip to interesting Louisiana spending one week to your profit and enjoyment on the beautiful campus of Louisiana State University.

For programs and other information write to: Dr. H. T. Karnes, Director, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana.

SYLLABUSES FOR COURSES IN MODERN NUMERICAL ANALYSIS

Graduate courses in modern numerical analysis are being given with the assistance of the National Bureau of Standards personnel at the American University, Washington, D. C. and at the University of California in Los Angeles. Syllabuses for these courses may be obtained by writing to: Chief of the Computation Laboratory, National Bureau of Standards, Washington 25, D. C., or to Director of Research, Institute for Numerical Analysis, National Bureau of Standards, U.C.L.A., Los Angeles 24, California.

THE JOURNAL OF RATIONAL MECHANICS AND ANALYSIS

The publication of a new journal entitled *Journal of Rational Mechanics and Analysis* was announced recently. The Editors request that subscriptions and inquiries regarding the *Journal* be sent to them at the following address: Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana. The subscription price is $18.00 per volume; for those engaged in teaching or research the rate is $6.00 per volume.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1952:

Boston University. May 26 to July 5: Professor Scheid, numerical methods; Professor Johanson, differential equations. July 7 to August 16: Professor Giever, elementary theory of numbers; Dr. Browder, introduction to modern mathematics; Dr. Pennington, Cambridge University, divergent series.

New York University, School of Education. June 30 to August 8: Dr. Dodes, application of mathematics, teaching aids in mathematics education; Professor Kinsella, current trends in mathematics curricula, testing and evaluation in mathematics education.

Northwestern University. June 20 to August 23: Lecturer Wyler, non-Euclidean geometry; Professor Boothby, introduction to analytic topology, engineering mathematics II; Professor Springer, differential equations, functions of a complex variable; Professor Wescott, theory of statistics; Professor Zelinsky, algebra of matrices and quadratic forms, introduction to modern algebra; Professor Scott, definite integrals, seminar in analysis; Professor Simmons, theory of equations; Professor Davis, history and teaching of mathematics, nonlinear problems of the physical sciences; Professor Moulton, engineering mathematics I.

University of California, Berkeley, Department of Mathematics. June 23 to August 2: Professor McDonald, coordination in mathematics for secondary schools.

University of California, Berkeley, Statistical Laboratory. June 23 to August 2: Professors Neyman and Fix and Mr. LeCam, statistical methods of searching for causal relationships; Dr. Kallianpur, elements of probability and statistics; Professor Neyman, individual research leading to higher degrees. August 4 to September 13: Professor Bates, second course in probability and statistics, applications of mathematical statistics; Professor Neyman, individual research leading to higher degrees.

University of Illinois. June 13 to August 9: Staff, differential equations, advanced calculus, fundamental concepts of mathematics, introduction to higher geometry, introduction to higher analysis, measure and integration, mathematical methods in engineering and science; Professor Atchison, algebraic geometry; Professor Bourgin, topological spaces; Professor Brahana, linear algebra; Professor Ketchum, theory of functions of a complex variable.

University of Nebraska. June 2 to July 25: Professor Doole, advanced Euclidean geometry; Professor Camp, elementary differential equations; Professor Ribeiro, solid analytic geometry; Professor Basoco, topics in analysis.

University of South Carolina. June 10 to August 12: Professor Lytle, statistics, theory of equations; Professor Weber, college geometry; Professor Hedberg, higher algebra; Professor Williams, functions of a complex variable.

PERSONAL ITEMS

Professor J. E. Davis, Drexel Institute of Technology, and Professor Marguerite Lehr of Bryn Mawr College were appointed to serve as delegates of the Association at the meeting of the American Academy of Political and Social Science which was held in Philadelphia, Pennsylvania, on April 18–19, 1952.

Professor J. M. Earl of the University of Omaha represented the Association at the inauguration of President C. H. Canfield of Tarkio College on March 27, 1952.

Brooklyn College announces the following: Assistant Professor A. W. Landers has been promoted to an associate professorship; Mr. Leon Herbach has been appointed to an instructorship.

Gonzaga University reports: Assistant Professor D. R. Ryan has been promoted to an associate professorship; Instructors J. J. Murray and G. M. Simoneau have been promoted to assistant professorships; Mr. J. A. Ryan, previously a student at the University, has been appointed to an instructorship.

Haverford College announces the following appointments: Assistant Professor R. C. James, formerly of the University of California at Berkeley, to an associate professorship; Dr. A. F. Strehler of the University of Chicago to an instructorship.

Kenyon College announces: Dr. D. T. Finkbeiner, formerly an instructor at Yale University, has been appointed to an associate professorship; Professor

W. R. Transue is on leave of absence and is in Italy on a Fulbright grant.

At Purdue University: Associate Professor Michael Golomb has been promoted to a professorship; Assistant Professor Sam Perlis has been promoted to an associate professorship; Mr. E. L. Klinger has been promoted to an assistant professorship; Dr. Meyer Jerison, previously research instructor at the University of Illinois, Dr. J. H. B. Kemperman, formerly research worker at Mathematisch Centrum, Amsterdam, Dr. C. R. Putnam, previously a member of the Institute for Advanced Study, and Assistant Professor H. E. Teicher of the University of Delaware have been appointed to assistant professorships.

Union College announces the following: Dr. D. S. Morse, chairman of the Department of Mathematics, has been appointed to the recently established Marie Louise Bailey professorship in mathematics; Associate Professor C. T. Male, Sr., has been promoted to a professorship; Mr. W. C. Stone has been appointed to an assistant professorship.

United States Naval Postgraduate School, now located at Monterey, California, makes the following announcements: Associate Professors W. E. Bleick, Joseph Giarratana, A. B. Mewborn, and C. C. Torrance have been promoted to professorships; Assistant Professors F. D. Faulkner, Walter Jennings, and F. M. Pulliam have been promoted to associate professorships; Dr. T. E. Oberbeck, formerly with the Operations Evaluation Group, Division of Industrial Cooperation, Massachusetts Institute of Technology, has been appointed Associate Professor of Mathematics and Mechanics.

At the University of California at Berkeley: Associate Professor A. L. Foster has been promoted to a professorship; Assistant Professor E. L. Lehmann, who is on leave of absence during 1951–52, has been promoted to an associate professorship; Instructors Evelyn Fix and Elizabeth L. Scott have been promoted to assistant professorships; Dr. Jan Kalicki and Dr. A. V. Martin have been appointed to visiting assistant professorships; Dr. Gopinath Kallianpur has been appointed Lecturer; Dr. Harley Flanders has been appointed to an instructorship; Professor B. A. Bernstein has retired with the title of Professor Emeritus; Assistant Professors S. P. Diliberto and J. L. Hodges are on leave of absence during 1951–52.

University of Chicago announces the following: Dr. Erwin Kleinfeld, previously an assistant at the University of Wisconsin, has been appointed to an instructorship; Professor Andre Ammann of the University of Geneva, Switzerland, Dr. George Austin, previously National Research Council Fellow, Dr. Leila A. Dragonette of the University of Pennsylvania, Professor Lino Gutierrez of the University of Habana, Mr. Mischa Cotlar, Guggenheim Fellow from the University of Buenos Aires, and Dr. Israel Herstein of the Cowles Commission have been appointed Research Associates in Mathematics; Associate Professor P. R. Halmos is on a year's leave of absence and is spending it at the University of Montevideo, Institute of Mathematics and Statistics, as Visiting Lecturer under the auspices of the United States Department of State; Professor M. H. Stone is Visiting Lecturer for the American Mathematical Society during 1952–

53; Assistant Professor I. E. Segal is on a year's leave of absence and is on a Guggenheim Fellowship at the Institute for Advanced Study; Assistant Professor E. H. Spanier was at the Institute for Advanced Study during the Autumn Quarter of 1951.

University of Cincinnati announces the promotion of Associate Professor Meyer Salkover to a professorship and the promotions of Assistant Professors J. L. Baker and W. E. Restemeyer to associate professorships.

At the University of Kentucky: Instructor M. S. Davis has been promoted to an assistant professorship; Graduate Assistants S. B. Vanaman, Jr., and W. M. Zaring have been promoted to part-time instructorships; Mr. H. W. Burnette of Alabama Polytechnic Institute and Mr. J. B. Wells, Jr., have been appointed to part-time instructorships.

University of Tennessee announces the appointments of Mr. H. S. Davis and Mr. R. D. Sheffield to teaching assistantships.

University of Wisconsin makes the following announcements: Dr. C. W. Curtis, previously a graduate student at Yale University, and Dr. E. R. Immel have been appointed to instructorships.

Mr. L. J. Abbeduto is teaching in the Hardin County Schools, Kenton, Ohio.

Associate Professor G. F. Allison of the University of New Brunswick is now at Ryerson Institute, Toronto, Canada.

Dr. E. L. Arnoff has a position as Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Dr. Sholom Arzt of New York University has a position as Mathematician at the Applied Physics Laboratory of Johns Hopkins University, Silver Spring, Maryland.

Assistant Professor Miriam C. Ayer of Wellesley College has received an appointment as Mathematician at Sandia Corporation, Albuquerque, New Mexico.

Dr. H. G. Ayre, professor of mathematics and director of the General College Division at Western Illinois State College, has been promoted to Head of the Department of Mathematics and is continuing in the position as Director of the General College Division.

Assistant Professor Frederick Bagemihl has resigned from the University of Rochester.

Mr. L. E. Berg is teaching in Fairmount, Indiana.

Assistant Instructor R. N. Bradt of the University of Kansas has been appointed to a graduate assistantship at Stanford University.

Mr. Robert Bryant, formerly an instructor at Miami University, has accepted a position at the Naval Research Laboratory, Washington, D. C.

Mr. C. M. Callahan, formerly a teaching fellow at Alabama Polytechnic Institute, has been appointed to an instructorship at Lincoln Memorial University.

Mr. R. E. Carroll of Emory University has a position as Dynamics Engineer at Bell Aircraft Corporation, Niagara Falls, New York.

Mrs. Virginia L. Chatelain has been appointed to an instructorship at Adela Hale Business Career School, Hutchinson, Kansas.

Professor Emeritus A. B. Coble of the University of Illinois has been appointed Visiting Professor at the University of North Carolina for the winter and spring quarters of 1951–52.

Mr. D. L. Daly of the University of Kentucky has been appointed to an instructorship at the University of Maine.

Mr. D. J. Ewy has been appointed to an instructorship at Fresno State College.

Dr. W. T. Fishback of the University of Vermont has been promoted to an assistant professorship.

Mr. M. P. Guhsé, formerly a graduate student at the University of Massachusetts, is teaching in St. Lucie County High School, Ft. Pierce, Florida.

Professor Vaclav Hlavaty of Indiana University has been appointed to the staff of the Graduate Institute for Applied Mathematics of the University.

Mr. J. S. Hokanson of Ohio University has been appointed to an instructorship at Ripon College.

Dr. J. H. Hornback has been appointed to an assistant professorship at the University of Alabama.

Mr. D. B. Houghton has been promoted to the position of Acting Chief of the Analysis Section, Franklin Institute, Philadelphia, Pennsylvania.

Acting Professor S. W. Howell of Yankton College has been promoted to the position of Professor of Mathematics and Physics and Head of Department of Mathematics.

Mr. D. B. Jordan, formerly a teaching fellow at Polytechnic Institute of Brooklyn, has a position as Junior Engineer at Sylvania Electric Products, Incorporated, Bayside, New York.

Mr. R. J. Lambert of Iowa State College has a position as Mathematician in the Defense Department, Washington, D. C.

Dr. D. H. Lehmer, formerly professor of mathematics at the University of California at Berkeley, has been appointed Director of Research of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, California.

Miss Margaret M. Linney, formerly a student at Regis College, is now Engineering Aide at the General Electric Company, Schenectady, New York.

Dr. W. G. Lister has been awarded a post-doctoral AEC Fellowship and has been spending the academic year 1951–52 at Yale University.

Assistant Professor R. W. Long of Washington and Jefferson College has been appointed Lecturer at the University of Pittsburgh.

Dr. Nathaniel Macon, who was a Fulbright Fellow at the University of Amsterdam, the Netherlands, during 1950–1951, has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Mr. Paul Meier has accepted a position at the Forrestal Research Center, Princeton, New Jersey.

Mr. D. F. Mela has received an appointment as Mathematician in the Operations Evaluation Group, Navy Department, Washington, D. C.

Mr. A. J. Mortola, formerly a tutor at City College of the City of New York, is teaching at St. Peter's College, Jersey City, New Jersey.

Associate Professor F. C. Mosteller of Harvard University has been promoted to a professorship.

Professor S. B. Myers of the University of Michigan is serving as Acting Head of the Department of Mathematics during the second semester of 1951–52 while Professor T. H. Hildebrandt is on leave of absence.

Associate Professor P. B. Norman of Wagner Memorial Lutheran College has accepted a position as Project Engineer, Curtiss-Wright Corporation, Carlstadt, New Jersey.

Miss Frances A. Norton, formerly an Instructor at Alabama Polytechnic Institute, has a position with the Hayes Aircraft Corporation, Birmingham, Alabama.

Dr. F. D. Parker of Case Institute of Technology has been appointed to an assistant professorship at St. Lawrence University.

Mr. V. E. Roberts, formerly a mathematics teacher in Fairbanks High School, Alaska, has been appointed to an instructorship at the University of Alaska.

Mr. G. B. Robinson, previously a teaching fellow at Cornell University, has been appointed to an instructorship at the University of Connecticut.

Mr. L. L. Ross of Ohio Northern University has accepted a position with Babcock and Wilcox Company, Barberton, Ohio.

Miss Jean E. Sammet, formerly an assistant at the University of Illinois, has a position as a computer for the Metropolitan Life Insurance Company, New York City.

Mr. R. C. Seber of Rockford College has been appointed to an assistant professorship at Wisconsin State College.

Mr. H. A. Seebald has a position as Methods Engineer with Remington Rand, Incorporated, New York City.

Assistant Professor Edward Silverman of Kenyon College is a staff member of Sandia Corporation, Albuquerque, New Mexico.

Mr. Sam Stallard, formerly lecturer at the University of British Columbia, has been appointed Lecturer at the University of New Brunswick.

Instructor Irwin Stoner of the University of Minnesota is now Assistant Scientist at the Rosemount Research Center of the University.

Instructor R. T. Tear of Rensselaer Polytechnic Institute has been recalled to active service in the United States Air Force.

Instructor J. G. Wendel of Yale University has accepted a position with the Rand Corporation, Santa Monica, California.

Mr. H. W. Wessely, previously a student at Illinois Institute of Technology,

has a position as Physicist with the Gaertner Scientific Corporation, Chicago, Illinois.

Assistant Professor G. M. Wing of the University of California at Los Angeles has been on leave of absence during 1951–52 and has been a staff member of the Los Alamos Scientific Laboratory, New Mexico.

Mr. R. G. Woodle, Jr., of the Missouri School of Mines and Metallurgy has been appointed to an instructorship at the University of Kansas.

Dr. R. A. Worsing of Iowa State College has joined the Physical Science Division of Boeing Airplane Corporation, Seattle, Washington.

Dr. F. H. Young, who was an AEC Research Fellow at the University of Oregon, is now at the Bureau of Ordnance, United States Navy, Washington, D. C.

Professor Emeritus W. B. Fite of Columbia University died on March 1, 1952. He was a charter member of the Association.

Professor F. G. Graff of Oberlin College died on February 22, 1952.

Dr. C. O. Lampland of the Lowell Observatory died on December 14, 1951. He was a charter member of the Association.

Professor V. S. Lawrence, Jr., of Virginia Polytechnic Institute died on February 20, 1952.

Professor Emeritus C. L. Poor of Columbia University died on September 27, 1951.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE EARLE RAYMOND HEDRICK LECTURES

Professor Tibor Rado of the Ohio State University will deliver a series of three expository lectures entitled "Derivatives and Jacobians" as the EARLE RAYMOND HEDRICK LECTURES at the meeting of the Association in East Lansing, Michigan on September 1 and 2, 1952. These lectures have been named in honor of Professor Earle Raymond Hedrick, one of the three organizers of the Association and its first President.

The Board of Governors has established the EARLE RAYMOND HEDRICK LECTURES in an effort to promote expository lectures and publications as significant activities of the Association.

Professor Rado's first lecture will be given at 4 P.M. on Monday, September 1. This lecture, to be devoted to a treatment of Jacobians, is intended to be an exposition which will be readily understood by anyone who is familiar with

advanced calculus and vector analysis as they occur in undergraduate courses.

The second lecture will be given at 9 A.M. on Tuesday, September 2. The first half of this lecture will be a continuation of the first lecture. The second half will be a description of relative Cech cohomology groups as the essential topological tool that is needed in the n-dimensional case. Professor Rado states that he has found a simple treatment of cohomology theory, and he hopes through these lectures to introduce this theory to the analysts for use in their work.

The third lecture will be given at 7:30 P.M. on Tuesday, September 2. This lecture, on a more advanced level than the other two, will be concerned with the transformation theory of multiple integrals in terms of the generalized Jacobians treated in the other two lectures.

Professor Rado states that the material of the three lectures will be published as a monograph of about two hundred pages. The manuscript will be ready for the printer in January, 1953.

The EARLE RAYMOND HEDRICK LECTURES for 1952 have been arranged by a special committee consisting of G. C. Evans, J. C. Oxtoby, and G. B. Price, chairman. At a later date this committee will recommend to the Board of Governors of the Association policies which will govern these lectures in future years.

PREPARATION OF PROBLEM AND SOURCE MATERIALS FOR THE MATHEMATICAL TRAINING OF SOCIAL SCIENTISTS

As readers of this MONTHLY probably know, a Committee on the Mathematical Training of Social Scientists has been at work for some time. The Committee includes representatives of twelve learned societies including the Mathematical Association of America.

As the result of a suggestion from this Committee, the Social Science Research Council is now sponsoring a small group to work during the summer of 1952. This group will attempt to compile from the literature of the various social sciences lists of problems, extracts from sources, and references to sources that illustrate varieties of uses of mathematics in the social sciences. These compilations are expected to serve a number of important ends—e.g., to provide mathematicians with material for use in texts and courses designed for social scientists, to indicate the general dimensions of the mathematical training appropriate for students of the social sciences now and in the future, and to facilitate the study of mathematics by social scientists for whom organized courses are not available.

This Committee believes that the group referred to would find it most helpful if it could start with a wide variety of suggestions from the various areas concerned. A general appeal for such suggestions is hereby made. They should be sent to Professor William G. Madow, Chairman, Committee on the Mathematical Training of Social Scientists, University of Illinois, Urbana, Illinois, as soon as possible.

Although the Committee does not wish to limit the suggestions to specific types of material, it would prefer greater emphasis on materials relating to the use of mathematics in the social sciences themselves than on those relating to statistics, since the materials necessary for statistics are better known. Finally, the Committee would appreciate learning where programs of mathematical training intended for social scientists are now in existence or in process of development, and where mathematics at the level of the calculus or higher is required for undergraduate or graduate degrees in the social sciences or may be substituted for another requirement for a degree in a social science.

THE MAY MEETING OF THE INDIANA SECTION

The twenty-eighth annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Saturday, May 5, 1951. Two sessions were held at which Professor H. E. Crull of Butler University, Chairman of the Section, presided.

There were fifty-five in attendance including the following forty members of the Association:

W. C. Arnold, W. L. Ayres, Juna L. Beal, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, K. W. Crain, H. E. Crull, M. W. De Jonge, W. E. Edington, P. D. Edwards, J. L. Ericksen, R. A. Gambill, E. L. Godfrey, Michael Golomb, S. H. Gould, G. H. Graves, J. R. Hadley, J. F. Heyda, M. W. Keller, E. L. Klinger, Jacob Korevaar, P. J. McCarthy, Gladys B. McColgin, P. T. Mielke, J. T. Montgomery, P. W. Overman, M. O. Peach, J. C. Polley, Arthur Rosenthal, Sister Gertrude Marie, J. D. Speas, R. B. Stone, Anna K. Suter, A. D. Talkington, Jane A. Uhrhan, F. J. Wagner, M. S. Webster, K. P. Williams, H. E. Wolfe.

The following officers were elected: Chairman, M. W. Keller, Purdue University; Vice-Chairman, Florence Long, Earlham College; Secretary-Treasurer, J. C. Polley, Wabash College.

The chairman announced the election of Professor J. C. Polley of Wabash College to the Board of Governors as Sectional Governor for the three year period July 1, 1951, to June 30, 1954.

Professor P. D. Edwards, Chairman of the Committee on Awards, reported that the committee had awarded five Association medals, four for high mathematical achievement in the Indiana science talent search and one to the top ranking contestant in the final comprehensive contest in mathematics of the Indiana high school achievement program sponsored by Indiana University.

The annual meeting of 1952 will be held at Indiana University, Bloomington, Indiana, at a date to be announced later.

The following papers were presented:

1. *The geometry of ideal gas flows*, by Mr. J. L. Ericksen, Indiana University.

The Mach number, m, of a stationary, isoenergetic ideal gas flow in the plane was shown to satisfy a cubic equation of the form $\sum_{n=0}^{n=3} f_n(m^2-1)^n$, where the functions f_n are expressible in terms of the adiabatic exponent, the curvature of the streamlines, the curvature of the orthogonal trajectories of the streamlines, and certain derivatives of these curvatures. It is possible to obtain analytic conditions to determine whether or not a given congruence of curves can be streamlined for such a flow.

2. *Continuation of the sequence addition, multiplication, exponentiation, . . .*, by Professor S. H. Gould, Purdue University.

Since multiplication is addition with equal summands and exponentiation is multiplication with equal factors, it is natural to define a fourth operation, namely, exponentiation with equal exponents, and then a fifth, and so forth. The higher operations have been neglected on the ground that they are neither associative nor commutative. In the present note these properties were replaced by corresponding inequalities, leading to generalization of some familiar theorems. A compact notation was devised and unsolved problems were suggested. It was pointed out that the notation makes it easy to write numbers which are strikingly large.

3. *Q scores, a correlation study*, by Sister Gertrude Marie, Marian College.

The aim of this paper was to evaluate the mathematics scores of ten consecutive classes of college women in the National Sophomore Testing Program as a measure of mathematical ability and/or achievement. These scores were correlated with: (1) *Q* scores and *I.Q.*'s as given by the American Council of Education Psychological Examination; (2) science, reading comprehension, and total scores on the Cooperative General Culture and English Tests of the National Sophomore Testing Program; (3) high school and college mathematics achievement as indicated by number and kind of courses taken and class marks received.

4. *Linear dependence and the Wronskian*, by Professor M. S. Webster, Purdue University.

The purpose of this paper was to discuss the theorem "The functions of a set are linearly dependent if, and only if, their Wronskian is identically zero." Although the theorem is in general false, it is found in several texts including two published in the period 1940–1950. Peano and Bôcher recognized that the theorem is not valid and Bôcher published several correct theorems related to the false one. Theorems of this type and a necessary and sufficient condition involving the Gramian were discussed.

5. *Student and teacher*, by Professor G. H. Graves, Purdue University.

Since specialization implies cooperation, mathematicians and educators should work in close collaboration. The graduate student is particularly the victim of misunderstanding between these two groups. The relation between teacher and student is a personal one, not purely professional. The teacher must be profoundly interested in the development of the student's capabilities. When a student fails, the teacher fails with him. Suggestions were made regarding use of class meetings, conferences, psychological aids to study, administration of tests, and the subject of grading. The opinion that the teacher's objectives should include making himself unnecessary was expressed.

6. *The use of a fourth property of a right triangle in teaching mathematics*, by Professor E. L. Godfrey, Defiance College.

The following theorem is a natural addition to the corollaries concerning the lengths of the sides and the perpendicular to the hypotenuse of a right triangle which are included in the standard textbook presentation of Euclid's eighth proposition of book six. Corollary: The rectangle contained by the sides of a right triangle is equal to the rectangle contained by the base and the perpendicular drawn from the vertex of the right angle to the base. The formula for the altitude suggested by this property was shown to be useful frequently in college mathematics. Since the proof is obvious from a consideration of areas instead of similar triangles, it was suggested as a suitable and useful exercise to introduce in junior high school courses in arithmetic, algebra and/or general mathematics.

7. *On the zeros and asymptotic behavior of Bessel functions*, by Professors

Michael Golomb and Jacob Korevaar, Purdue University, and presented by Professor Korevaar.

In this paper completely elementary methods were used to derive the common results concerning the real and non-real zeros and the asymptotic behavior of Bessel functions; some new results were given on the location of the non-real zeros. No particular representations of the Bessel functions by power series, definite integrals or otherwise were used. For each problem a form of Bessel's differential equation was used which best suited the specific problem. The results then followed readily from the definition of Bessel functions as solutions of the differential equation and were therefore applicable to any (real-valued) solution of this equation, whether it be a Bessel function of the first kind, of the second kind, or a linear combination of such functions.

8. *Newtonian pattern of analysis*, by Professor M. O. Peach, University of Notre Dame.

The concepts of Newtonian mechanics are often extended to phenomena which are essentially non-mechanical. When this is done it should be done consciously and with precision if pitfalls are to be avoided. The author raised the general question: To what extent and under what conditions can non-mechanical phenomena be analyzed according to the pattern of Newtonian mechanics? Precise conditions were laid down for the introduction of the concept of force. A generalized definition of force, applicable to both mechanical and non-mechanical phenomena, was given. Necessary and sufficient conditions for the validity of the parallelogram law for adding forces and of Newton's third law were deduced. An application of these ideas to the theory of dislocations was briefly sketched. The incomplete nature of the answer given to the above question was emphasized, and several directions in which research of potential mathematical interest could be undertaken were pointed out.

9. *A theorem on improper integrals in abstract spaces*, by Professor P. T. Mielke, Wabash College.

In this paper the notion of improper integral is that used by H. Hahn and A. Rosenthal in *Set Functions*, the University of New Mexico Press, 1948. The theorem proved is one of several contained in the author's doctoral thesis, *Improper Integrals in Abstract Spaces*. The space E is to be considered a general space without any particular topological or metric structure; M a σ-field of subsets of E, $\phi(M)$ a totally additive set function in M which is assumed complete for ϕ. The function $f(x)$ is a point function, ϕ-measurable on a set $A \in M$. Theorem: If $\phi(A)$ is infinite and $f(x)$ is ϕ-summable on A, then, given $\epsilon > 0$, there exists a $\bar{\delta} > 0$ such that for every δ-scale $\{z_i\}$ with $\delta \leqq \bar{\delta}$

$$\left| L(f, \phi, \{z_i\}, A_\delta) - (A) \int f d\phi \right| < \epsilon$$

where $A_\delta = A[|f(\hat{x})| > \delta]$ and $L(f, \phi, \{z_i\}, A_\delta)$ is any of the four Lebesgue sums for f, ϕ, $\{z_i\}$, and A_δ.

Professor J. B. Irwin of the Department of Astronomy of Indiana University delivered by invitation an address entitled *The Possibilities of the Electronic Computer in Astronomical Research*. In this he discussed the nature and construction of recently developed computers and presented certain problems in astronomy to illustrate the possible use of the electronic computer in astronomical research.

J. C. POLLEY, *Secretary*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The fifth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the State College of Washington, Pullman, Washington, on June 15, 1951, in conjunction with the four hundred seventy-first meeting of the American Mathematical Society. Professor A. F. Moursund, Chairman of the Section, invited Professors R. M. Winger and R. D. James to share the duties of presiding at the afternoon session.

There were fifty persons in attendance, including the following thirty-six members of the Association:

H. A. Antosiewicz, T. M. Apostol, R. W. Ball, J. P. Ballantine, R. A. Beaumont, J. S. Biggerstaff, Z. W. Birnbaum, J. L. Botsford, J. L. Brenner, L. G. Butler, Harold Chatland, P. A. Clement, K. L. Cooke, C. M. Cramlet, D. B. Dekker, Howard Eves, K. S. Ghent, J. W. Green, F. L. Griffin, Mary E. Haller, I. M. Hostetter, H. H. Irwin, R. D. James, J. M. Kingston, M. S. Knebelman, R. B. Leipnik, J. J. Livers, A. T. Lonseth, R. E. Lowney, L. H. McFarlan, A. F. Moursund, D. C. Murdoch, Ivan Niven, L. J. Paige, J. R. Vatnsdal, R. M. Winger.

A business meeting was held in the evening at which the following officers were elected: Chairman, Professor R. D. James, University of British Columbia; Vice-Chairman, Professor A. S. Merrill, University of Montana; Secretary-Treasurer, Professor J. M. Kingston, University of Washington.

It was decided to hold the next meeting of the Section at the University of Oregon, Eugene, Oregon, on June 20, 1952, and to make it, if possible, a joint meeting with the Society. A committee consisting of Professors Murdoch (Chairman), Eves, and Leipnik was appointed to investigate the possibility and advisability of conducting mathematical contests among high school students in this area. Professors Niven (Chairman) and Brenner, and Dr. Ball were appointed as a Program Committee to organize next year's program in the direction of providing greater usefulness to high school teachers. Additional problems in mimeographed form were made available by the Committee on a Problem Book in Mathematics.

The afternoon session consisted of the following eight fifteen minute papers and an invited hour address.

1. *The separation of a proper fraction into two unit fractions*, by Professors Howard Eves, Oregon State College, and H. Ohbayashi, Nagoya University, Japan. Presented by Professor Eves.

This paper summarizes much existing information on this interesting problem and considers some applications to conjectures concerning the method of solution of the problem by the ancient Egyptians. A proof is given of a result of Ohbayashi to the effect that a proper fraction p/q, $(p, q) = 1$, is separable into two unit fractions if and only if there exists an integer x such that $x \mid q'$ and $x^2 \equiv -q' \pmod{p}$, where $q' \mid q$. Distinct values of x lead to different unique separations.

2. *Remarks on definition and notation used in calculus*, by Professor R. B. Leipnik, University of Washington.

Although clear, correct definitions and consistent notations are available for the concepts introduced in elementary calculus, they are often avoided. Some unconventional definitions and no-

tations for functions, derivatives, differentials, etc., are described, and their advantages discussed, with the hope that they might be more widely adopted.

3. *Equivalence relations and canonical forms*, by Professor R. A. Beaumont and Dr. W. R. Ball, University of Washington. Presented by Dr. Ball.

The authors suggest that the idea of an equivalence relation on a set S and the corresponding partitioning of S should be emphasized as a fundamental algebraic concept in upper division algebra courses. A study of groups of transformations comes early in such a course, and since every group of transformations on a set S defines an equivalence relation, the latter concept may be introduced in a natural way at that time. The discussion of the representatives of equivalence classes leads to a simple and precise explanation of what is meant by canonical forms. Topics such as the reduction of matrices to canonical forms, congruence of integers, cosets of a group, and residue classes of a ring are unified by a careful study of equivalence relations, and the study itself can be used as an example of the process of abstraction. Numerous examples of equivalence relations from elementary mathematics may be given which clarify the earlier material and explain the concept to the student.

4. *Some recent work in the theory of discrete groups*, by Professor J. L. Brenner, State College of Washington.

This is an expository summary of some definitions and results.

5. Invited address: *On estimating a probability distribution from a finite number of observations*, by Professor Z. W. Birnbaum, University of Washington.

The probability distribution of a random variable X is known as soon as the distribution function $F(x) = \text{Prob}\{X \leqq x\}$ is given. In practical applications it often happens that $F(x)$ is not known, but one has the means to obtain a set of N independent observations (a sample of size N) of X. The practical problem then arises: is it possible to utilize the information contained in such a finite sample to obtain an estimate of $F(x)$? In 1933, A. Kolmogorov gave a rather surprising answer to this question. His results initiated a number of further publications, and the present paper is a report on developments in this field, including some very recent findings.

6. *Transversal families of curves*, by Professor L. H. McFarlan, University of Washington.

The transversality condition of the calculus of variations in the plane is applied to pairs of one parameter families of curves in order to obtain a generalization of the idea of orthogonal families of curves. Given a one parameter family of curves, an integrand function is found such that each member of the given family is transversal to every member of a second one parameter family of curves.

7. *Modular invariants associated with resistance networks*, by Professor C. M. Cramlet, University of Washington.

A graph of V vertices, and B connecting branches, supplies a topological space S for a finite geometry. The chain, circuit, coordinate system (C S) are defined geometric objects. A proper (C S) is a set of $N = B - V + 1$ linearly independent proper circuits (single closed loops). Any two (C S) are related by a transformation matrix $t \in T$ having elements $0, 1, -1$ and determinant ± 1. Products of the elements of T generate a group G, matrices of integers with determinant ± 1. Additional structure is endowed S by associating a resistance matrix $r_{\alpha\beta}$ and mesh currents i^{α} with a

preferred ($C\,S$) and extending those to extended ($C\,S$) of G by the assumption that these are covariant and contravariant respectively. *A property of S* is defined by the vanishing of a representation of G. In a similar manner a geometry may be defined by replacing G by the matrices of G reduced modulo 2. The symmetric tensor $r_{\alpha\beta}$, $N=2$, has a complete set of three independent invariants,

$$I = r_{11} + r_{12} + r_{22}, \qquad J = r_{12}^2 + r_{11}r_{22}, \qquad K = r_{12}(r_{12} + r_{11})(r_{12} + r_{22}).$$

For $N=3$ an invariant of degree 2, two of degree 3 and three of degree 5 constitute a complete set.

8. *A basis for pseudo-vectors*, by Professor I. M. Hostetter, Oregon State College.

Let A and B be points of an euclidean 3-space and a and b real scalars. The sum $aA+bB$, $a+b\neq 0$, is defined after Grassmann as the point C of weight $a+b$ which divides the segment AB in the ratio $b:a$. $A+(-1)B=A-B$ is defined as the free line vector BA, $A\times B$ as the glissant bound to the line through A and B. The outer product of a glissant and a free vector, $(A\times B)\times(C-B)=(A\times B)\times C$, is defined as an areal vector bound to the plane through $A\times B$ which is parallel to $C-B$.

If G_1, G_2, G_3 are three glissants of unit length forming an equilateral triangle, any glissant U in the plane may be written in terms of this basis as $U=4/3\epsilon^{ijk}p_iq_jG_k$, where p_i and q_j are the signed perpendicular distances from the sides of the triangle to the initial and terminal points of U in any position on its line of action. Similarly, if B_1, B_2, B_3, B_4 are the outward drawn areal vectors bound to the plane of and determined by the faces of a tetrahedron whose edges are of unit length, any areal vector W bound to a plane may be expressed in terms of the basis B_1 by $W=3/4\sqrt{6}\ \epsilon^{hijk}p_hq_ir_jB_k$, where p_h, q_i, r_j are the signed perpendicular distances from the planes of the tetrahedron to the vertices of any triangular representation of W.

9. *The reduction formulas for* $\int \sin^m\theta\cos^n\theta\,d\theta$, by Professor J. P. Ballantine, University of Washington.

The reduction formulas for $\int s^m c^n d\theta$ are rendered almost symmetrical, and hence more easily memorized, by the introduction of

$$\mu = -(m+n+1).$$

$$\int s^m c^n d\theta = \pm\frac{s^{m'}c^{n'}}{D} + \frac{N}{D}\int s^{m+\Delta m}c^{n+\Delta n}d\theta,$$

where $(\Delta m, \Delta n, \Delta\mu)$ is some permutation of $(+2, -2, 0)$. Hence 6 formulas. $m'=1+$smaller of $(m, m+\Delta m)$, $n'=1+$smaller of $(n, n+\Delta n)$. The sign of the term $s^{m'}c^{n'}/D$ is the same as the sign of $\Delta m-\Delta n$. $D=m+1$ if $\Delta m=2$, $D=n+1$ if $\Delta n=2$, $D=-(\mu+1)$ if $\Delta\mu=2$. $N=m-1$ if $\Delta m=-2$, $N=n-1$ if $\Delta n=-2$, $D=-(\mu-1)$ if $\Delta\mu=-2$.

If one of the parameters, m, n, or μ, is odd and positive, make a substitution in which $\cos\theta$, $\sin\theta$, or $\tan\theta$, respectively, is a new variable. If all the positive parameters are even, reduce them by applying the reduction formulas.

The reduction formulas fail when $D=0$, that is, when the parameter being increased starts at -1. Hence they fail when (m, n, μ) is any permutation of $(1, -1, -1)$ or $(-1, 0, -0)$. In the case of $(1, -1, -1)$, a substitution can be made.

The rules for the reduction formulas for $\int \sinh^m t\cdot\cosh^n t\,dt$ are the same, except that the sign before $s^{m'}c^{n'}/D$ is always plus.

J. MAURICE KINGSTON, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday, October 12, 1951. Professor C. M. Pirrong, Chairman of the Section, presided.

Sixty-four persons attended the meeting, including the following thirty-nine members of the Association:

E. F. Allen, R. V. Andree, Arthur Bernhart, J. C. Brixey, Sarah M. Burkhart, R. L. Caskey, Y. W. Chen, N. A. Court, R. B. Deal, R. C. Dragoo, H. F. Fehr, B. T. Goldbeck, A. A. Grau, E. V. Greer, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. E. Heimann, W. N. Huff, L. W. Johnson, J. T. Krattiger, G. D. Kyle, J. E. LaFon, Eunice Lewis, H. W. Linscheid, R. D. McDole, Dora McFarland, G. E. Meador, R. R. Murphy, C. M. Pirrong, D. P. Richardson, Herbert Scholz, H. W. Smith, T. C. Smith, C. E. Springer, Vivian Spurgeon, G. R. Vick, J. H. Zant.

At the business session the following officers were elected: Chairman, L. W. Johnson, Oklahoma A. and M. College; Vice-Chairman, Truman Wester, Central State Teachers College; Secretary, R. V. Andree, University of Oklahoma.

The program consisted of the following five papers.

1. *On the incenter of a triangle*, by Professor E. F. Allen, Oklahoma A. and M. College.

This paper studied the projection of the incenter of a triangle under a projection which carried the circumscribing circle of the triangle into a central conic.

2. *Geometric figures and p-rings*, by Professor A. A. Grau, University of Oklahoma.

The p-rings may be considered as a generalization of Boolean algebras, which may be identified with 2-rings. The structure theorems of Stone and McCoy-Montgomery were reviewed and the following properties of a finite p-ring R were discussed: The elements x of R may be represented by n-tuples $(\xi_1, \cdots, \xi_n)$ of elements ξ_i in the residue class ring of the integers mod p. Let $\phi(\xi) = m$, the integer such that $0 \leqq m < \frac{1}{2}(p-1)$ and either $m \equiv \xi$ or $m \equiv \rho - \xi \pmod{p}$. Then $\rho(x,y) \equiv \sum_{i=1}^{n} \phi(x_i - y_i)$ is a metric over R. An *edge* is a pair of points x, y such that $\rho(x, y) = 1$. The metric thus associates with R a geometric configuration consisting of points and edges. The property of 2-rings, that given any 3 points x_1, x_2, x_3 there exists a unique point x where $f(x) = \sum_{i=1}^{3} \rho(x, x_i)$ takes on its minimum value, does not hold in a p-ring where p is odd for three given points or, more generally, for n given points.

3. *Property of points whose pedal triangles are similar*, by Professor G. D. Kyle, Arkansas State A. M. and N. College.

By a rotational process a clinant was developed and a series of points were found having pedal triangles similar to a given triangle. It was shown that one of these points was the circumcenter of the given triangle and the others were vertices of the first and second Brocard triangles of the given triangle. The pedal triangles of the inverses of the above points, excepting the circumcenter, were shown to be similar to the original triangle.

4. *On improving the teaching of college mathematics*, by Professor J. H. Zant, Oklahoma A. and M. College.

The task of improving the teaching of college mathematics was attacked from the point of view of the job facing the college teacher of mathematics.

The college teacher of mathematics must (1) have a broad, well-rounded education, (2) must understand young people and know how to direct their learning activities, (3) must be able to select, organize the subject matter of the field of mathematics at many levels and for many purposes, (4) must be able to evaluate the results of his efforts, (5) must be willing and able to engage in some sort of creative activity, and finally, (6) must assume the obligations of an educated citizen in the college community, the state, and the world.

This job calls for preparation in the areas of general education, mathematical education and professional education and a definite effort should be made by the undergraduate and graduate schools and by departments of mathematics to see that prospective college teachers have access to courses which will insure such knowledge and skills. If final answers to these problems exist, they must be worked out by the professors in the departments of mathematics and in the graduate schools where the prospective teachers are being trained and where the beginner starts his work.

5. *Mathematics in the English secondary school and college*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

Professor Hamilton, who has recently returned from an exchange professorship in England, discussed instruction in mathematics in the English secondary school and university. He called attention to the methods by means of which only the best students are selected to attend the university and to the differences between the English and American practices with respect to examinations and the granting of degrees.

J. C. BRIXEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, May 3, 1952.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.
MISSOURI, Lindenwood College, St. Charles, May 2, 1952.
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October 31, 1952.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Delaware, Newark, November 29, 1952.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May 23–24, 1952.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.
WISCONSIN, University of Wisconsin in Milwaukee, May 10, 1952.

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CONTENTS

The General Chinese Remainder Theorem . . . Oystein Ore 365
The Sums of the Dihedral and Trihedral Angles in a Tetrahedron . . . . . J. W. Gaddum 370
Some Properties of Continuous Functions . . . M. K. Fort, Jr. 372
The Unsatisfactory Story of Curvature . . . . J. L. Coolidge 375
A Canonical Basis for the Ideals of a Polynomial Domain. . . . . G. Szekeres 379
Mathematical Notes . . . . . L. Carlitz, J. G. Campbell, . . . Leonard Tornheim, Richard Bellman, I. Opatowski 386
Classroom Notes. . . . . M. K. Fort, Jr., C. G. Phipps, ... H. J. Hamilton, Editorial, Corrections 392
Elementary Problems and Solutions 407
Advanced Problems and Solutions 412
Recent Publications 421
Clubs and Allied Activities 423
News and Notices 427
The Mathematical Association of America 431
New Members 431
December Meeting of the Maryland-D.C.-Virginia Section . . 434
November Meeting of the Philadelphia Section 436
October Meeting of the Minnesota Section 438
January Meeting of the Northern California Section 440

JUNE-JULY 1952